AN EXPOSITION ON GELFAND-FUKS COHOMOLOGY

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Contents

1. Introduction

Beginning around fifty years ago, a plethora of literature has been created to understand the continuous Lie algebra cohomology $\mathfrak{X}(M)$ of the vector fields on a smooth manifold M. This cohomology in the literature carries the name *Gelfand-Fuks cohomology*, in reference to the authors who opened the investigation of this subject with a series of highly novel papers [1], [2], [3]. Initially, it was hoped that this cohomology might contain invariants for the smooth structure of M, hence be a potential tool for understanding the problem of classifying and differentiating exotic smooth structures on a given manifold, a problem which is still open as of today, for example for the 4-sphere.

Unfortunately, these hopes were denied by a paper by Bott and Segal, which showed that the Gelfand-Fuks cohomology was isomorphic to the singular cohomology of a mapping space that can be functorially constructed from M, and from which no new invariants arise [4]. Regardless, these explorations brought with them a lot of applications, for example in the theory of foliations [5] or for the construction of the Virasoro algebra [6]. Further, a lot of related open problems are still being pursued, like the continuous cohomology of the Lie algebra of symplectic, Hamiltonian or divergence-free vector fields on symplectic/Riemannian manifolds [7] [8].

To this end, we want to lay out in this document a streamlined, detailed and relatively elementary path to the fundamental results of Gelfand-Fuks cohomology, guided by the general strategies in [9] and [10], filling in nontrivial details that have been left to the reader in the original literature, and modernizing some of the language used. We make neither a claim to originality – in fact, except for the occasional auxiliary Lemma, all given results are recorded in the literature – nor to be fully exhaustive – we restrict ourselves to Gelfand-Fuks cohomology with trivial coefficients, and direct the reader to [11] for an overview of the study of other coefficient modules.

Our final goal is to, in full detail, formulate and prove the existence of spectral sequences which calculate the Gelfand-Fuks cohomology of certain smooth manifolds, in other words, to reproduce [9, Theorem 2.4.1a, 2.4.1.b]. This document is intended to be fully accessible for any researcher with a solid understanding of the basics of homological algebra, sheaf theory and differential geometry.

We begin in Section 2 with an outline of the continuous cohomology of the Lie algebra of formal vector fields, i.e. vector fields whose coefficient functions are formal power series. They represent the infinitesimal counterpart of $\mathfrak{X}(M)$ and their cohomology can be calculated exactly, using a spectral sequence over which one can get full control. In Section 3, we tie the cohomology of formal vector fields to the Gelfand-Fuks cohomology of Euclidean space, which may itself be understood as the local counterpart to Gelfand-Fuks cohomology. We do not only calculate this cohomology, but also examine its transformation behaviour under diffeomorphisms of Euclidean space. This prepares a local-to-global analysis of the global Gelfand-Fuks cohomology on an arbitrary smooth manifold. In Section 4, by patching together the local Gelfand-Fuks cohomology using sheaf theoretic ideas, we end up being able to give a variation of the well-known spectral sequences that calculate Gelfand-Fuks cohomology for a class of orientable, smooth manifolds. While it is difficult to get good control over this spectral sequence for arbitrary manifolds, we explain how it allows a full calculation of the Gelfand-Fuks cohomology of the circle S^1 and may be used to make certain general statements about finite-dimensionality of the Gelfand-Fuks cohomology. While these results in this section are well-known, our proof is a novel addition to the currently available ones, exploiting the existence of so-called *k-good covers*, see [12], an approach inspired by the treatment of Gelfand-Fuks cohomology in the framework of factorization algebras in the preprint [13]. This proof allows an analogous treatment of other infinite-dimensional Lie algebras such as gauge algebras, as we plan to show in future work [14].

2. The Lie algebra of formal vector fields

In this section, we mainly elaborate on the methods given in [9, Chapter 2.2] and [3]. There will be one substantial divergence in method in the proof of Theorem 2.24, which makes use of the results of [15], see Subsection 2.2.

2.1. Definition and first properties.

We begin with an analysis of the infinitesimal counterpart of the vector fields on smooth manifold.

Definition 2.1 (Formal vector fields)**.** Fix any point $p \in M$. We define the *Lie algebra of formal vector fields* W_n to be equal to the Lie algebra of ∞-jets $J_p^\infty \mathfrak{X}(M)$ at p.

Any choice of coordinates $(x_1, \dots, x_n)$ centered at p with local frame $(\partial_1, \dots, \partial_n)$ allows us to write W_n as

$$W_n = \left\{ \sum_{i=1}^n f_i \partial_i : f_i \in \mathbb{R}[\![x_1, \dots, x_n]\!] \right\} \cong \mathbb{R}[\![x_1, \dots, x_n]\!] \otimes \mathbb{R}^n,$$

with induced Lie bracket

$$[f\partial_i, g\partial_j] := f\frac{\partial g}{\partial x_i} \cdot \partial_j - g\frac{\partial f}{\partial x_j} \cdot \partial_i, \quad f, g \in \mathbb{R}[\![x_1, \dots, x_n]\!].$$

Equipping $\mathbb{R}[\![x_1, \dots, x_n]\!]$ with its projective limit topology, W_n becomes a topological Lie algebra.

A quick argument on why the notation W_n does not reflect on the choices made.

Lemma 2.2. *Denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M.*

Let $U, V \subset M$ be open neighbourhoods and $p \in U$. A diffeomorphism

$$\phi : U \to V$$

induces a Lie algebra isomorphism

$$j^\infty \phi : J_p^\infty \mathfrak{X}(M) \to J_{\phi(p)}^\infty \mathfrak{X}(M)$$

which only depends on the ∞-jet of ϕ at p.

In particular, the definition of W_n is independent of the choice of basepoint p, up to Lie algebra isomorphism.

Proof. A choice of a different point q and coordinates around q corresponds to a local diffeomorphism $\phi : U_p \to U_q$ of small neighbourhoods U_p and U_q of p and q, respectively, only depending on the infinity-jet of ϕ at p. Since Lie brackets of vector fields commute with pullbacks by diffeomorphisms, these pullbacks are Lie algebra isomorphisms.

As a result, since all smooth manifolds are locally Euclidean, our definition of W_n depends only on the dimension n, up to Lie algebra isomorphism.

□

Remark 2.3. Later on, we will compare copies of W_n at different points of a given manifold, and hence we have to stay aware about the fact that W_n transforms as a Lie algebra of jets. This transformation behaviour will induced geometric effects in the continuous Lie algebra cohomology of vector fields which are not apparent in the formal case.

We first examine the structure of W_n. The following lemma is a standard verification and will be presented without proof.

Lemma 2.4. *As a topological Lie algebra, the formal vector fields is the completion of a graded vector space* $W_n := \widehat{\bigoplus}_{k\geq -1}\mathfrak{g}_k$ *with*

$$\mathfrak{g}_k := \left\{ \sum_{i=1}^n p_i\partial_i \in W_n : p_i \text{ homogeneous polynomials of degree } k+1 \right\}.$$

This grading is compatible with the Lie algebra structure, meaning $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ *for all* $i, j \geq -1$.

For all $X \in W_n \setminus \{0\}$, write $\deg X = k$ if $X \in \mathfrak{g}_k$. If $X \in W_n$ lies in one of the $\mathfrak{g}_k$, we say X is *homogeneous.*

In low orders, we have Lie algebra isomorphisms:

$$\mathfrak{g}_{-1} = \operatorname{span}\{\partial_i : i = 1, \ldots, n\} \cong \mathbb{R}^n, \quad \mathfrak{g}_0 = \operatorname{span}\{x_i\partial_j : i, j = 1, \ldots, n\} \cong \mathfrak{gl}_n(\mathbb{R}).$$

Definition 2.5. The element $E := \sum_{i=1}^n x_i\partial_i \in \mathfrak{g}_0$ is the *Euler vector field* of W_n.

Remark 2.6. The Euler vector field E allows an alternative definition of the grading on W_n:

$$\mathfrak{g}_k = \{X \in W_n : [E, X] = k \cdot X\}.$$

This naturally motivates our choice of grading and why $\partial_1, \ldots, \partial_n \in W_n$ are considered to be of negative degree.

Definition 2.7. Let $\mathfrak{g}$ be a topological Lie algebra. The *Chevalley-Eilenberg cochain complex* of $\mathfrak{g}$ is

$$C^\bullet(\mathfrak{g}) := \bigoplus_{k\geq 0} C^k(\mathfrak{g}),$$

where $C^k(\mathfrak{g})$ is the space of multilinear, skew-symmetric, jointly continuous maps $c : \mathfrak{g}^k \to \mathbb{R}$.

It is equipped with the differential

$$d : C^k(\mathfrak{g}) \to C^{k+1}(\mathfrak{g}),$$

$$dc(X_1, \ldots, X_{k+1}) = \sum_{1\leq i<j\leq k+1} (-1)^{i+j-1} c([X_i, X_j], X_1, \ldots, X_{k+1}). \tag{2.1}$$

The cohomology of this complex is denoted $H^\bullet(\mathfrak{g})$, called the *continuous Chevalley-Eilenberg cohomology.*

Remark 2.8. Note that the degree zero differential equals the zero map.

Remark 2.9. If $\mathfrak{g}$ is finite-dimensional, the continuity assumption is redundant.

If $\mathfrak{g} = W_n$ with its projective topology, then $c \in C^\bullet(W_n)$ just means that c is only nonzero on a finite-dimensional subspace of $\Lambda^\bullet W_n$, i.e. for all cochains c there is a $k \in \mathbb{Z}$ so that $c(X, \cdot, \ldots, \cdot) = 0$ for all X with $\deg X > k$.

Recall the following:

Definition 2.10. Let $\mathfrak{g}$ be a Lie algebra, $Y \in \mathfrak{g}$ and $c \in C^k(\mathfrak{g})$.

i) Denote the natural Lie algebra action of an element Y on $C^k(\mathfrak{g})$ by $Y \cdot c$; the formula is given for $Y, X_1, \ldots, X_k \in \mathfrak{g}$ by

$$(Y \cdot c)(X_1, \ldots, X_k) := \sum_{i=1}^k c(X_1, \ldots, [Y, X_i], \ldots, X_k).$$

ii) Denote by $Y \lrcorner c \in C^{k-1}(\mathfrak{g})$ the *interior product of c with Y*, which is defined via

$$(Y \lrcorner c)(X_1, \ldots, X_{k-1}) = c(Y, X_1, \ldots, X_{k-1}).$$

We have the following homotopy relation between interior product and differential:

Lemma 2.11. *Let $\mathfrak{g}$ be a Lie algebra, $c \in C^\bullet(\mathfrak{g})$ and $Y \in \mathfrak{g}$. Then we have the following chain homotopy formula:*

$$d(Y \lrcorner c) + Y \lrcorner dc = Y \cdot c.$$

Proof. We have

$$\begin{aligned}
d(Y \lrcorner c)(X_1, \ldots, X_k) &= \sum_{i<j} (-1)^{i+j-1} (Y \lrcorner c)([X_i, X_j], X_1, \ldots \widehat{i} \ldots \widehat{j} \ldots, X_k) \\
&= \sum_{i<j} (-1)^{i+j-1} c(Y, [X_i, X_j], X_1, \ldots \widehat{i} \ldots \widehat{j} \ldots, X_k) \\
&= \sum_{i<j} (-1)^{i+j} c([X_i, X_j], Y, X_1, \ldots \widehat{i} \ldots \widehat{j} \ldots, X_k), \\
Y \lrcorner (dc)(X_1, \ldots, X_k) &= dc(Y, X_1, \ldots, X_k) \\
&= \sum_{i<j} (-1)^{i+j-1} c([X_i, X_j], Y, X_1, \ldots \widehat{i} \ldots \widehat{j} \ldots, X_k) \\
&\quad + \sum_{i=1}^{k} (-1)^{k+1} c([Y, X_i], X_1, \ldots \widehat{i} \ldots, X_k).
\end{aligned}$$

Adding the two terms gives us the desired equality. □

A nice, well-known corollary of the previous statement is:

Corollary 2.12. The action of a Lie algebra $\mathfrak{g}$ on its cohomology $H^\bullet(\mathfrak{g})$ is trivial.

Using the grading of W_n induced by the Euler vector field E, we can also define a grading of the cochains:

Definition 2.13. Let $r \in \mathbb{Z}$ and $k > 0$. We define $C^k_{(r)}(W_n) \subset C^k(W_n)$ as the subspace of cochains c with the following property: for all homogeneous $X_1, \ldots, X_k \in W_n$ we have

$$\sum_{i=1}^{k} \deg X_i \neq r \implies c(X_1, \ldots, X_k) = 0.$$

We further set $C^0_{(r)}(W_n) = \begin{cases} C^0(W_n) & \text{if } r = 0, \\ 0 & \text{else.} \end{cases}$

Proposition 2.14 ([9], Section 1.5 and 2.2)**.** The spaces $C^\bullet_{(r)}(W_n)$ fulfil the following properties:

i) We have

$$C^\bullet(W_n) = \bigoplus_{r \in \mathbb{Z}} C^\bullet_{(r)}(W_n),$$

and for all $r \in \mathbb{Z}$, the spaces $C^\bullet_{(r)}(W_n)$ are subcomplexes of $C^\bullet(W_n)$.

ii) If $r < -k$, then $C^k_{(r)}(W_n) = 0$.

iii) The inclusion $C^\bullet_{(0)}(W_n) \subset C^\bullet(W_n)$ induces an isomorphism

$$H^\bullet(C^\bullet_{(0)}(W_n)) \cong H^\bullet(W_n).$$

In the following we will write $H^\bullet_{(r)}(W_n) := H^\bullet(C^k_{(r)}(W_n))$.

Proof. i) The direct sum decomposition follows since every $c \in C^k(W_n)$ is only nonzero on a finite-dimensional subspace, and its evaluation on any $X_1, \dots, X_k \in W_n$ can be uniquely decomposed into summands of homogeneous vector fields.

For all homogeneous $X_a, X_b \in W_n$, we either have $[X_a, X_b] = 0$ or $\deg[X_a, X_b] = \deg X_a + \deg X_b$. In the former case, every cochain vanishes on $[X_a, X_b]$. In the latter case, if $\sum_{i=1}^{k+1} \deg X_i \neq r$, then for all $1 \leq a, b \leq k+1$ we have

$$\deg[X_a, X_b] + \sum_{i \neq a,b} \deg X_i = \sum_{i=1}^{k+1} \deg X_i \neq r.$$

This implies $dC^\bullet_{(r)}(W_n) \subset C^\bullet_{(r)}(W_n)$, hence the $C^\bullet_{(r)}(W_n)$ are subcomplexes.

ii) Due to the pidgeonhole principle, any collection of k elements in W_n whose degrees sum up to a value smaller $-k$ must have an element with degree smaller -1. Such an element is necessarily zero, which shows the statement.

iii): Let $r \neq 0$. Then, Lemma 2.11 yields the following homotopy equation on $C^\bullet_{(r)}(W_n)$:

$$d(E \lrcorner c) + E \lrcorner (dc) = r \cdot c.$$

As such, the map $\frac{1}{r}(E \lrcorner \cdot)$ defines a chain homotopy between the identity and zero for the cochain complex $C^\bullet_{(r)}(W_n)$, and hence $H^\bullet(C^\bullet_{(r)}(W_n)) = 0$ for all $r \neq 0$.

We conclude that all cohomology classes of $C^\bullet(W_n)$ admit a representative fully contained $C^\bullet_{(0)}(W_n)$, which shows the required statement. □

2.2. **Stable cohomology of W_n.** We first focus on certain low-dimensional cohomology, the so-called *stable cohomology* of W_n, due to Guillemin and Shnider. They prove in [15, Corollary 1] that $H^k(W_n)$ is trivial in dimension $k = 1, \dots, n$. Note that their paper makes much more general statements, in particular about stable cohomology of formal Lie algebras corresponding to other classical vector field Lie algebras, e.g. formal Hamiltonian and divergence-free vector fields.

Definition 2.15. Define for all $r \in \mathbb{Z}$,

$$\partial C^\bullet_{(r)}(W_n) := \{\partial_i \cdot c \in C^\bullet_{(r+1)}(W_n) : c \in C^\bullet_{(r)}(W_n)\}.$$

Recall that $\partial_i \cdot c$ denotes the action of $\partial_i \in \mathfrak{g}_{-1}$ on the cochain c (see Definition 2.10).

Lemma 2.16. *For all $r \in \mathbb{Z}$, the space $\partial C^\bullet_{(r)}(W_n)$ is a subcomplex of $C^\bullet_{(r+1)}(W_n)$.*

Proof. It suffices to prove $\partial_i \cdot dc = d(\partial_i \cdot c)$, which follows directly from Lemma 2.11. □

We need one more preparing definition, since the component of degree zero in $C^\bullet(W_n)$ is often troublesome.

Definition 2.17 (Reduced Complex)**.** If $C^\bullet$ is a cochain complex, define the *reduced complex* $\tilde{C}^\bullet$ as

$$\tilde{C}^0 = 0, \quad \tilde{C}^k := C^k \quad \forall k \geq 1,$$

equipped with the inherited differential from $C^\bullet$. When notationally more feasible, we may also denote the reduced complex with an index $C^\bullet_{\text{red}}$.

We denote the cohomology of the reduced complex $\tilde{H}^\bullet := H^\bullet(\tilde{C}^\bullet)$.

Remark 2.18. For reduced Chevalley-Eilenberg cohomology we have

$$\tilde{H}^k(\mathfrak{g}) = H^k(\mathfrak{g}) \quad \forall k \geq 1.$$

The aim of this section is the construction of a Koszul complex relating the complexes $C^\bullet_{(r)}(W_n)$ for different values of r. To this end, let us first prove a more technical lemma:

Lemma 2.19. *Over the abelian Lie algebra $\mathfrak{g}_{-1}$, the continuous dual W_n^* is a* free *module, i.e. it is free as a module over the enveloping algebra $U(\mathfrak{g}_{-1}) \cong S^\bullet(\mathfrak{g}_{-1})$. This module structure extends to a free module structure on the reduced chain complex $\tilde{C}^\bullet(W_n)$ and is compatible with the cochain differential.*

Proof. Define for $j = 1, \dots, n$ the continuous functionals $\partial_j^* : W_n \to \mathbb{R}$ with

$$\partial_j^*(\partial_i) = \delta_{ij}, \quad \partial_j^*(x_{i_1} \dots x_{i_k} \cdot \partial_i) = 0, \quad 1 \leq i_1, \dots, i_k, i, j \leq n.$$

Let us show that the collection $B := \{\partial_1^*, \dots, \partial_n^*\} \subset W_n^*$ defines a basis of W_n^* with respect to the $\mathfrak{g}_{-1}$-module structure. Clearly, B is linear independent in W_n^*.

The Lie bracket of $\partial_i \in W_n$ with a formal vector field $X \in W_n$ equals the partial derivative of X along x_i, hence:

$$(\partial_{i_1} \dots \partial_{i_r} \cdot \partial_j^*)(x_{k_1} \dots x_{k_s} \partial_l) = \begin{cases} 1 \text{ if } x_{i_1} \dots x_{i_r} = x_{k_1} \dots x_{k_s} \text{ and } j = l, \\ 0 \text{ else.} \end{cases}$$

But since every element of W_n^* is only nonzero on a finite-dimensional subspace of W_n, this shows that B generates W_n^* as a $\mathfrak{g}_{-1}$-module. Hence B is a basis for W_n^* and W_n^* is free over $\mathfrak{g}_{-1}$. But then also the exterior product $\Lambda^k W_n^* = C^k(W_n)$ is free for all $k \geq 0$, and also $C^\bullet(W_n)$ and $\tilde{C}^\bullet(W_n)$.

The compatiblity of the module structure with the differential of $\tilde{C}^\bullet(W_n)$ is essentially Lemma 2.16. □

Proposition 2.20. There exists an exact sequence of cochain complexes

$$\begin{aligned} 0 \to \tilde{C}^\bullet_{(0)}(W_n) &\to \tilde{C}^\bullet_{(1)}(W_n) \otimes_{\mathbb{R}} \mathfrak{g}_{-1}^* \\ &\to \dots \\ &\to \tilde{C}^\bullet_{(n)}(W_n) \otimes_{\mathbb{R}} (\Lambda^n \mathfrak{g}_{-1})^* \\ &\to \left(\tilde{C}^\bullet_{(n)}(W_n) / \partial \tilde{C}^\bullet_{(n-1)}(W_n)\right) \otimes_{\mathbb{R}} (\Lambda^n \mathfrak{g}_{-1})^* \\ &\to 0, \end{aligned}$$

where the differentials in every term are induced by the Lie algebra differential of $\tilde{C}^\bullet(W_n)$.

Proof. Denote by S^+ the functor assigning to a vector space the space of symmetric tensors in nonzero degree. Then, consider first the well-known, acyclic Koszul complex

$$0 \to S^\bullet(\mathfrak{g}_{-1}) \to S^\bullet(\mathfrak{g}_{-1}) \otimes_\mathbb{R} \mathfrak{g}_{-1}^* \to \cdots \to S^\bullet(\mathfrak{g}_{-1}) \otimes_\mathbb{R} (\Lambda^n \mathfrak{g}_{-1})^* \\ \to S^\bullet(\mathfrak{g}_{-1})/S^+(\mathfrak{g}_{-1}) \otimes_\mathbb{R} (\Lambda^n \mathfrak{g}_{-1})^* \to 0.$$

Here, the last nontrivial map is given by the quotient map, and all other nontrivial maps are given by

$$\sigma_r : S^\bullet(\mathfrak{g}_{-1}) \otimes_\mathbb{R} (\Lambda^r \mathfrak{g}_{-1})^* \to S^\bullet(\mathfrak{g}_{-1}) \otimes_\mathbb{R} (\Lambda^{r+1} \mathfrak{g}_{-1})^*, \\ u \otimes v \mapsto \sum_{i=1}^n (\partial_i \cdot u) \otimes (\partial_i^* \wedge v) \quad \forall u \in S^\bullet(\mathfrak{g}_{-1}), v \in (\Lambda^r \mathfrak{g}_{-1})^* .$$

The action of $S^\bullet(\mathfrak{g}_{-1})$ commutes with these maps, i.e. for all $u, w \in S^\bullet(\mathfrak{g}_{-1}), v \in (\Lambda^\bullet \mathfrak{g}_{-1})^*$ we have

$$w \cdot \sigma_r(u \otimes v) = \sum_{i=1}^n (w \cdot \partial_i \cdot u) \otimes (\partial_i^* \wedge v) = \sigma_r(w \cdot u \otimes v),$$

and straightforwardly also with the quotient map $S^\bullet(\mathfrak{g}_{-1}) \to S^\bullet(\mathfrak{g}_{-1})/S^+(\mathfrak{g}_{-1})$.

Free modules are flat, and as such the tensor product of this complex with $\tilde{C}^\bullet(W_n)$ over the ring $S^\bullet(\mathfrak{g}_{-1})$ is still exact, thus we get an exact complex of vector spaces

$$\begin{aligned} 0 &\to \tilde{C}^\bullet(W_n) \\ &\to \tilde{C}^\bullet(W_n) \otimes_\mathbb{R} \mathfrak{g}_{-1}^* \\ &\to \dots \\ &\to \tilde{C}^\bullet(W_n) \otimes_\mathbb{R} (\Lambda^n \mathfrak{g}_{-1})^* \\ &\to (\tilde{C}^\bullet(W_n))/(\tilde{C}^\bullet(W_n) \otimes_{S^\bullet(\mathfrak{g}_{-1})} S^+(\mathfrak{g}_{-1})) \otimes_\mathbb{R} (\Lambda^n \mathfrak{g}_{-1})^* \to 0. \end{aligned} \tag{2.2}$$

By Lemma 2.19, the chain complex structure of $C^\bullet(W_n)$ is compatible with the action of $S^\bullet(\mathfrak{g}_{-1})$, and thus (2.2) is not only a complex of vector spaces but of chain complexes.

The maps σ_r of the original Koszul complex induce the following maps on Complex (2.2):

$$\tilde{C}^\bullet(W_n) \otimes (\Lambda^r \mathfrak{g}_{-1})^* \to \tilde{C}^\bullet(W_n) \otimes (\Lambda^{r+1} \mathfrak{g}_{-1})^*, \\ c \otimes v \mapsto \sum_{i=1}^n (\partial_i \cdot c) \otimes (\partial_i^* \wedge v) \quad \forall c \in \tilde{C}^\bullet(W_n), v \in \Lambda^r \mathfrak{g}_{-1}.$$

This shows, firstly, by exactness of (2.2) at the last term:

$$\tilde{C}^\bullet(W_n) \otimes_{S^\bullet(\mathfrak{g}_{-1})} S^+(\mathfrak{g}_{-1}) \cong \partial \tilde{C}^\bullet(W_n),$$

and secondly, that the complex (2.2) decomposes with respect to the grading of $\tilde{C}^\bullet(W_n) = \bigoplus_r \tilde{C}^\bullet_{(r)}(W_n)$, every differential increasing the degree $\tilde{C}^\bullet_{(r)}(W_n) \to \tilde{C}^\bullet_{(r+1)}(W_n)$.

Considering the component of the complex which starts with the term $\tilde{C}^\bullet_{(0)}(W_n)$ yields the desired statement. □

Proposition 2.21 ([15], Corollary 1)**.** We have $H^k(W_n) = 0$ if $k = 1, \dots, n$.

Proof. Consider the exact sequence from Proposition 2.20. We have established that $\tilde{C}^\bullet_{(r)}(W_n)$ is an acyclic complex whenever $r \neq 0$, and as such, all terms in the exact sequence are acyclic except for the leftmost and rightmost nontrivial ones. One can consider this exact sequence of cochain complexes as a double complex and examine the arising spectral sequences. The filtration by rows gives a spectral sequence that immediately collapses to zero due to exactness of the complex. On the other hand, the spectral sequence arising through filtration by columns can only converge to zero if the differentials on the n-th page induce the following isomorphisms:

$$H^k\left(\tilde{C}^\bullet_{(0)}(W_n)\right) \cong H^{k-n}\left(\left(\tilde{C}^\bullet_{(n)}(W_n)/\partial\tilde{C}^\bullet_{(n-1)}(W_n)\right) \otimes_{\mathbb{R}} (\Lambda^n \mathfrak{g}_{-1})^*\right).$$

But the complex on the right-hand side is zero in all degrees ≤ 0, hence so is its cohomology. Hence, for all $k = 1, \ldots, n$ we have

$$H^k(W_n) \cong H^k_{(0)}(W_n) = \tilde{H}^k_{(0)}(W_n) = 0.$$

□

2.3. **A spectral sequence for $H^\bullet(W_n)$.** One can do even better than Proposition 2.21: We will formulate a spectral sequence due to Gelfand and Fuks [3] which calculates the cohomology of W_n, and fully specify how the differentials in this spectral sequence map. In other words, the dimension of $H^\bullet(W_n)$ in every degree can be calculated for every $n \in \mathbb{N}$. The information from the previous section about low degree cohomology will aid us for the analysis of this spectral sequence.

Another important tool in understanding this spectral sequence will be the Borel transgression theorem for spectral sequences. To formulate it, let us first define some terminology.

Definition 2.22. Let $\{E^{p,q}_r, d_r\}_{r\geq 0}$ be a cohomological, first-quadrant spectral sequence. Denote by $\kappa^{r+1}_r : \ker d_r \to E^{\bullet,\bullet}_{r+1}$ the natural quotient map from cocycles of the r-th page differential d_r to the $r+1$-th page, and

$$\kappa^s_r = \kappa^s_{s-1} \circ \cdots \circ \kappa^{r+1}_r \quad \forall s > r,$$

where the domain of κ^s_r is defined inductively as all the $c \in E^{\bullet,\bullet}_{r+1}$ in the domain of κ^{s-1}_r so that $\kappa^{s-1}_r c \in \ker d_{s-1}$.

We call an element $c \in E^{p,0}_2$ *transgressive* if, for all r with $2 \leq r < p+1$, we have that c is in the domain of κ^r_2.

Intuitively, the transgressive elements in $E^{p,0}_2$ are the ones which "survive" until the very last moment: Only the differential $d_{p+1} : E^{p,0}_{p+1} \to E^{0,p+1}_{p+1}$, also called the *transgression*, can map it to something nontrivial.

By abuse of notation, we often denote an element in the domain of κ^s_r by the same symbol as its image under κ^s_r in the higher page $E^{\bullet,\bullet}_s$

The following theorem was originally proven in [16], but we cite a slightly stronger version from [17, Thm 2.9].

Theorem 2.23 (Borel transgression theorem). *Let $B^\bullet := \bigoplus_{p\in\mathbb{N}_0} B^p$ and $P^\bullet := \bigoplus_{q\in\mathbb{N}_0} F^q$ be finite-dimensional, graded vector spaces. Assume there are elements $x_i \in F^\bullet$ of odd degree such that*

$$\Lambda^\bullet[x_1, \ldots, x_l] \to F^\bullet$$

is bijective in degrees $\leq N$ and injective in degree $N+1$.

Let further $\{E_r^{p,q}, d_r\}_{r\geq 0}$ *be a cohomological spectral sequence whose second page has the shape*

$$E_2^{p,q} = B^p \otimes F^\bullet,$$

and which converges towards a graded vector space $H^\bullet$ *with* $H^k = 0$ *if* $0 < k \leq N+2$.

Then we can choose the generators x_i *to be transgressive, and if* $y_1, \dots, y_l \in B^\bullet$ *denotes a collection of elements with*

$$d_{\deg x_i + 1} x_i = y_i \quad i = 1, \dots, l,$$

then the map

$$\mathbb{R}[y_1, \dots, y_l] \to B^\bullet$$

is bijective for degrees $\leq N$ *and injective for degree* $N+1$.

Theorem 2.24 ([9], Theorem 2.2.4)**.** *Let* $n \geq 1$.

a) *There is a multiplicative, cohomological spectral sequence* $\{E_r^{p,q}, d_r\}$ *converging to* $H^\bullet(W_n)$ *with*

$$E_2^{0,\bullet} = \Lambda^\bullet[\phi_1, \phi_3, \dots, \phi_{2n-1}],$$
$$E_2^{\bullet,0} = \mathbb{R}[\Psi_2, \Psi_4 \dots, \Psi_{2n}]/\langle \Psi_{i_1} \dots \Psi_{i_k} : i_1 + \dots + i_k > 2n \rangle,$$
$$E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$$

where ϕ_i, Ψ_j *are multiplicative generators in degree* i *and* j *of the zeroeth column and row, respectively.*

b) *The differentials of the spectral sequence for* W_n *are fully specified on the generators by*

$$d_{i+1}\phi_i = \Psi_{i+1} \quad i \in \{1, 3, \dots, 2n-1\}.$$

c) *We have* $E_\infty^{p,q} = 0$ *if* $p \leq n$ *and* $(p,q) \neq (0,0)$, *or if* $p + q \leq 2n$.

Proof. Define $V := \mathbb{R}^n$ to simplify notation.

a) Consider the Hochschild-Serre spectral sequence of the pair $\mathfrak{g}_0 \subset W_n$ in continuous cohomology, see Appendix C. This has the first page

$$\begin{aligned} E_1^{p,q} &= H^q\left(\mathfrak{g}_0; \Lambda^p(W_n/\mathfrak{g}_0)^*\right) \\ &= H^q\left(\mathfrak{g}_0; \Lambda^p\left(\bigoplus_{j\neq 0} \mathfrak{g}_j\right)^*\right) \\ &= \bigoplus_{p_{-1}+p_1+p_2+\dots=p} H^q\left(\mathfrak{g}_0; \bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*\right) \end{aligned}$$

Note that, as Lie algebras, $\mathfrak{g}_0 \subset W_n$ is isomorphic to $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{gl}(V)$ via

$$\sum_{i,j=1}^n a_{ij} x_i \partial_j \mapsto (a_{ij})_{1\leq i,j\leq n}.$$

Also, in the above spectral sequence, the action of $\mathfrak{gl}_n(\mathbb{R}) \cong \mathfrak{g}_0$ on the finite-dimensional spaces $\bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*$ is an action on a tensor module, see Appendix B. By Theorem B.4, we may reduce the coefficient space in the above cohomologies to the $\mathfrak{gl}_n(\mathbb{R})$-invariants.

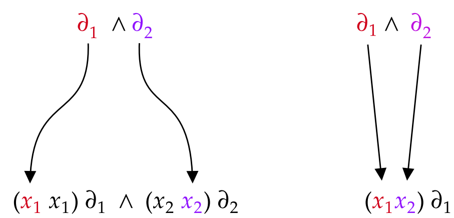

FIGURE 1. The arrows on the left-hand-side indicate a nontrivial contraction of tensor factors, whereas the right-hand-side contracts to zero, since the x_i commute with one another.

Hence,

$$E_1^{p,q} = \bigoplus_{p_{-1}+p_1+p_2+\cdots=p} H^q\left(\mathfrak{g}_0; \left(\bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*\right)^{\mathfrak{gl}(V)}\right)$$
$$= H^q(\mathfrak{g}_0) \otimes \left(\bigoplus_{p_{-1}+p_1+p_2+\cdots=p} \left(\bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*\right)^{\mathfrak{gl}(V)}\right).$$

By counting, we find that for any set of indices $p_{-1}, p_1, p_2, \ldots$, the amount of factors transforming *covariantly* under $\mathfrak{gl}(V)$ (i.e. copies of V^*) within $\bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*$ is equal to $p = p_{-1} + p_1 + \ldots$, whereas the amount factors transforming *contravariantly* (i.e. copies of V) is equal to $2p_1 + 3p_2 + \ldots$.

By Theorem B.2, there are only nonzero invariants in $\bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*$ if the p_j are chosen such that the amount of contravariant factors equals the amount of covariant factors.

Equivalently, this is the equation

$$p_{-1} = p_1 + 2p_2 + 3p_3 + \ldots \tag{2.3}$$

Simultaneously, again from Theorem B.2, we know that $\mathfrak{gl}_n(\mathbb{R})$-invariants in a tensor module

$$V^{\otimes r} \otimes (V^*)^{\otimes r} \cong \mathrm{Hom}((V^*)^{\otimes r} \otimes V^{\otimes r}, \mathbb{R})$$

can be described as the linear combinations of the functionals which contract all covariant indices with permutations of the contravariant indices.

Correspondingly, the $\mathfrak{gl}(V)$ invariants in the subspaces $\bigotimes_{j\neq 0} \Lambda^{p_j}\mathfrak{g}_j^*$ are given by subjecting these functionals to the required (skew-)symmetrizations.

Hence: If $p_{-1} > p_1 + p_2 + \ldots$, then, by the pidgeonhole principle, any invariant tensor contracts at least two contravariant factors belonging to $\Lambda^{p_{-1}}\mathfrak{g}_{-1}$ with two covariant factors, both belonging to a single copy of $\mathfrak{g}_j$ within $\Lambda^{p_j}\mathfrak{g}_j = \mathfrak{g}_j \wedge \cdots \wedge \mathfrak{g}_j$ for some $j \geq 1$ (compare Figure 2.3).

However, in such a contraction the contravariant factors would behave skew-symmetrically and the covariant ones symmetrically under permutation, hence their contraction is zero.

Thus, we get the additional requirement

$$p_{-1} \leq p_1 + p_2 + \dots \tag{2.4}$$

Combining (2.3) and (2.4) we end up with $p_k = 0$ for $k \geq 2$ and $p_{-1} = p_1 =: r$. This implies that $p = 2r$ is even whenever there are nontrivial invariants, and thus every other column in the page $E_1^{p,q}$ vanishes. Hence all differentials on the first page are trivial, and $E_1^{p,q} = E_2^{p,q}$.

Let us describe more explicitly the contractions that define the functionals in $(\Lambda^r \mathfrak{g}_{-1} \otimes \Lambda^r \mathfrak{g}_1)^{\mathfrak{gl}(V)}$. Our previous analysis implies that elements in this space arise by taking, for every permutation $\sigma \in \Sigma_r$, the functional

$$\bigotimes^r V \times \bigotimes^r (V^* \otimes V^* \otimes V) \to \mathbb{R},$$

1 with

$$\begin{aligned}(\alpha_1 \otimes \ldots \otimes \alpha_r, (\beta_1^1 \otimes \beta_1^2 \otimes \alpha_{r+1}) \otimes \cdots \otimes (\beta_r^1 \otimes \beta_r^2 \otimes \alpha_{2r}))\\ \mapsto \beta_1^1(\alpha_1) \ldots \beta_r^1(\alpha_r) \cdot \beta_1^2(\alpha_{r+\sigma(1)}) \ldots \beta_r^2(\alpha_{r+\sigma(r)}), \quad \forall \alpha_i \in V,\, \beta_i^1, \beta_i^2 \in V^*,\end{aligned}$$

skew-symmetrizing over the first r and the last r arguments, and symmetrizing over the exchange $\beta_i^1 \leftrightarrow \beta_i^2$. Denote the arising functional by $\Psi_\sigma \in (\Lambda^r \mathfrak{g}_{-1} \otimes \Lambda^r \mathfrak{g}_1)^{\mathfrak{gl}(V)}$.

By re-enumerating, one shows that Ψ_σ is, up to a sign, invariant under conjugation $\sigma \mapsto \tau\sigma\tau^{-1}$ for $\tau \in \Sigma_k$ and, using the inherited wedge product, we have for $\sigma \in \Sigma_r, \tau \in \Sigma_l$ and $r + l \leq n$:

$$\Psi_\sigma \wedge \Psi_\tau = \Psi_{\sigma\tau} \in \left(\Lambda^{r+l} \mathfrak{g}_{-1} \otimes \Lambda^{r+l} \mathfrak{g}_1\right)^{\mathfrak{gl}(V)}.$$

Because $\mathfrak{g}_{-1}$ is n-dimensional, $(\Lambda^r \mathfrak{g}_{-1} \otimes \Lambda^r \mathfrak{g}_1)^{\mathfrak{gl}(V)}$ is zero if $r > n$, so in particular products $\Psi_\sigma \wedge \Psi_\tau$ become zero if $\sigma \in \Sigma_r, \tau \in \Sigma_l$ and $r + l \leq n$.

Since every permutation can be decomposed into cycles, we can describe the invariants $(\Lambda^r \mathfrak{g}_{-1} \otimes \Lambda^r \mathfrak{g}_1)^{\mathfrak{gl}(V)}$ as the polynomial algebra in the generators $\Psi_{2r} \in (\Lambda^r \mathfrak{g}_{-1} \otimes \Lambda^r \mathfrak{g}_1)^{\mathfrak{gl}(V)}$ for every $r = 1, \ldots, n$, each Ψ_r corresponding to the conjugation class of the r-cycle in Σ_r, with the additional relation that products $\Psi_{i_1} \ldots \Psi_{i_k}$ are zero if $i_1 + \cdots + i_k > 2n$.

Summarizing:

$$E_2^{\bullet,q} = E_1^{\bullet,q} = H^q(\mathfrak{gl}_n(\mathbb{R})) \otimes \mathbb{R}[\Psi_2, \ldots, \Psi_{2n}]/\langle \Psi_{i_1} \ldots \Psi_{i_k} : i_1 + \cdots + i_k > 2n \rangle.$$

The cohomology of $\mathfrak{gl}_n(\mathbb{R})$ is calculated in Theorem B.5 to be equal to the exterior algebra $\Lambda^\bullet[\phi_1, \phi_3, \ldots, \phi_{2n-1}]$ with some multiplicative generators ϕ_i of degree i.

This proves part a) of the theorem.

b) Consider the above spectral sequence for the pair $(W_{3n}, \mathfrak{gl}_{3n}(\mathbb{R}))$. By Proposition 2.21, we know that $H^k(W_{3n}) = 0$ for $k = 1, \ldots, 3n$, and by part a) we know that the zeroeth column of the spectral sequence is equal to $\Lambda^\bullet[\phi_1, \phi_3, \ldots, \phi_{2n-1}]$ up to degree $2n$, the ϕ_i being the generators of $H^\bullet(\mathfrak{gl}_{3n}(\mathbb{R}))$. Hence we can apply the Borel transgression theorem 2.23 with $N = 2n-1$, implying that $\phi_1, \ldots, \phi_{2n-1}$ can be chosen so that

$$d_{2i}\phi_i = \Psi_{i+1}, \quad i \in \{1, 3, \ldots, 2n-1\}.$$

Now the inclusion $W_n \to W_{3n}$ induces a morphism from the spectral sequence for $(W_{3n}, \mathfrak{gl}_{3n}(\mathbb{R}))$ to the spectral sequence for $(W_n, \mathfrak{gl}_n(\mathbb{R}))$. Under this morphism, the generators $\phi_i \in H^\bullet(\mathfrak{gl}_{3n}(\mathbb{R}))$ restrict to equivalent generators in $\mathfrak{gl}_n(\mathbb{R})$ by Theorem

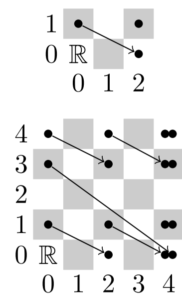

FIGURE 2. The spectral sequences for W_1 and W_2, with nonvanishing differentials indicated. Every dot represents one basis element of the term in the given position. The cohomology of W_1 is only nontrivial in degree 0 and 3, whereas the cohomology of W_2 is nontrivial in degree $0, 5, 7$ and 8, degree 5 and 8 having multiplicity 2.

B.5, and the generators Ψ_i for W_{3n} restrict to equivalent ones for W_n by the explicit formula for them given in part a).

This proves that all generators $\phi_1, \dots, \phi_{2n-1}$ in the spectral sequence for W_n map as desired. Since the differential of the Hochschild-Serre spectral sequence is multiplicative and all pages are generated by the ϕ_i and the Ψ_i, this fully specifies the differential on every page.

c) Any element in $E_2^{p,q}$ with $(p,q) \neq (0,0)$ is a linear combation of terms of the form

$$\phi_{i_1} \dots \phi_{i_s} \Psi_{j_1}^{m_1} \dots \Psi_{j_t}^{m_t},$$

where we have ordered the groups of indices ascendingly, i.e. $i_1 < \dots < i_s$ and $j_1 < \dots < j_t$, and s and t are possibly zero, but not both at the same time.

Part b) shows:

- If $s = 0$ or $i_1 > j_1$, the term is mapped to by $\phi_{j_1-1}\phi_{i_1} \dots \phi_{i_s} \Psi_{j_1}^{m_1-1} \dots \Psi_{j_t}^{m_t}$ by the differential $d_{2(j_1-1)}$.
- If $t = 0$ or $i_1 < j_1$, the term maps to $\phi_{i_2} \dots \phi_{i_s} \Psi_{i_1+1} \Psi_{j_1}^{m_1} \dots \Psi_{j_t}^{m_t}$ by d_{2i_1}.

Hence the proof is done if we can show that the product $\Psi_{i_1+1}\Psi_{j_1}^{m_1} \dots \Psi_{j_t}^{m_t}$ is nonzero if $p \leq n$ or $p + q \leq 2n$.

This is equivalent to showing

$$i_1 + 1 + m_1 j_1 + \dots + m_t j_t \leq 2n.$$

If $p \leq n$, then this is implied as follows:

$$i_1 < j_1 \leq m_1 j_1 + \dots + m_t j_t = p,$$
$$\implies i_1 + 1 + m_1 j_1 + \dots + m_t j_t \leq 2p \leq 2n.$$

On the other hand, assume $p + q \leq 2n$. Note that $s = 1$ is never the case, since i_1 is always odd and q is always even. Hence assume $s > 1$, so that $i_1 + 1 \leq i_1 + \dots + i_s$. But then

$$i_1 + 1 + m_1 j_1 + \dots + m_t j_t \leq i_1 + \dots + i_s + m_1 j_1 + \dots + m_t j_t = q + p \leq 2n.$$

This concludes the proof of c), and thus the theorem is shown. □

This allows one to fully calculate the dimensions of $H^\bullet(W_n)$ in all degrees and even offers some insight into the behaviour of representatives of the cohomology classes.

We are going to sumarize the most important properties of $H^\bullet(W_n)$ in the following corollary:

Corollary 2.25. The space $H^k(W_n)$ is finite-dimensional for all k, and trivial when $1 \leq k \leq 2n$ or $k > n^2 + 2n$. The wedge product of two cohomology classes of positive degree is zero.

3. Gelfand-Fuks cohomology on $\mathbb{R}^n$

In this section, we calculate the Gelfand-Fuks cohomology $H^\bullet(\mathfrak{X}(M))$ for Euclidean space $M = \mathbb{R}^n$. We follow the more elaborate outline by Bott, see [10]. The reader who is only interested in the calculation of $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ itself may skip to Remark 3.13 for a presentation of the considerably shorter proof from [9, Section 2.4.B, Lemma 1].

However, Bott's approach will allow us to easily extend our proof to the Gelfand-Fuks cohomology of a finite disjoint union $\bigsqcup_{i=1}^k \mathbb{R}^n$ and also to certain *diagonal* cohomologies thereof, a concept which we introduce in Section 4.

3.1. **Definitions and calculation.** Consider now again a smooth manifold M with the Lie algebra of smooth vector fields $\mathfrak{X}(M)$. This is a locally convex Lie algebra with respect to the standard Fréchet topology, and as such we can consider its continuous Chevalley-Eilenberg cohomology. We begin with an analysis of the *local* case $M = \mathbb{R}^n$.

There, we can express vector fields in the canonical coordinates

$$\mathfrak{X}(\mathbb{R}^n) = \left\{ \sum_{i=1}^n f_i \cdot \partial_i : f_i \in C^\infty(\mathbb{R}^n) \right\}$$

We also sometimes call $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ the *local Gelfand-Fuks cohomology*. We will relate the vector fields on $\mathbb{R}^n$ to the formal vector fields $W_n \cong J_0^\infty T\mathbb{R}^n$ by effectively contracting them to zero. Let us again identify some structures:

Definition 3.1. The subspace

$$\mathfrak{X}_{\text{pol}}(\mathbb{R}^n) := \left\{ \sum_{i=1}^n f_i \cdot \partial_i : f_i \in \mathbb{R}[x_1, \ldots, x_n] \right\} \subset \mathfrak{X}(\mathbb{R}^n)$$

is called the space of *polynomial* vector fields. They admit a filtration $P_0 \subset P_2 \subset \cdots \subset \mathfrak{X}_{\text{pol}}(\mathbb{R}^n)$ via

$$P_k := \{X \in \mathfrak{X}_{\text{pol}}(\mathbb{R}^n) : X = \sum_{i=1}^n f_i \partial_i \text{ with} \deg f_i = k+1 \text{ for all nonzero } f_i\},$$

where $\deg f$ denotes the polynomial degree of $f \in \mathbb{R}[x_1, \ldots, x_n]$. We write $\deg X = k$ if $X \in P_k$, and any X contained in $P_k \setminus P_{k-1}$ is called *homogeneous of degree* k.

Remark 3.2. The attentive reader will notice that this is in strong analogy to what we did with formal vector fields, and the grading of $W_n := \widehat{\bigoplus}_{k \geq -1} \mathfrak{g}_k$.

Indeed, the sets P_k are the images of the natural embeddings $\mathfrak{g}_k \to \mathfrak{X}(\mathbb{R}^n)$ that we get from considering finite formal vector fields in W_n as polynomial vector fields on $\mathbb{R}^n$. We will make use of this point of view later.

Definition 3.3. Let $t > 0$. We define:

i) The *scaling of* $\mathbb{R}^n$:
$$T_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \frac{x}{t},$$

ii) The *scaling of vector fields* as the pushforward of T_t:
$$(T_t)_* : \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n), \quad X \mapsto \frac{1}{t} \cdot (X \circ T_{t^{-1}}),$$

iii) For all $k \geq 1$, the *scaling of cochains* as the pullback of $(T_t)_*$:
$$T_t^* : C^k(\mathfrak{X}(\mathbb{R}^n)) \to C^k(\mathfrak{X}(\mathbb{R}^n)),$$
$$(T_t^* c)(X_1, \ldots, X_k) := c((T_t)_* X_1, \ldots, (T_t)_* X_k).$$

This next part of our proof is an elaboration of a step in Bott's lecture notes. Just like in Section 2, the tilde over a complex denotes a reduced complex.

Definition 3.4. For all $k \in \mathbb{Z}, q \geq 0$, define
$$F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n)) := \left\{ c \in \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n)) : \lim_{t \to 0} \frac{1}{t^k} T_t^* c(X_1, \ldots, X_q) \text{ exists } \forall X_i \in \mathfrak{X}(\mathbb{R}^n) \right\}.$$

Lemma 3.5. *The spaces $F^k \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ for $k \in \mathbb{Z}$ constitute a descending filtration of the chain complex $\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$.*

For $k \leq -n$ we have $F^k \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) = \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$, and
$$F^k \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \wedge F^l \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \subset F^{k+l} \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)).$$

Proof. Since the scaling of vector fields $(T_t)_*$ is a pushforward of a diffeomorphism of $\mathbb{R}^n$, it holds that
$$(T_t)_*[X, Y] = [(T_t)_* X, (T_t)_* Y] \quad \forall X, Y \in \mathfrak{X}(\mathbb{R}^n).$$

Hence, the scaling of cochains commutes with the Lie algebra differential, so if the appropriate limits exist for a cochain c, they also do for dc. Thus the $F^k \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ are indeed subcomplexes.

The filtration is descending since if $\lim_{t \to 0} \frac{1}{t^k} f(t)$ exists for some function $t \mapsto f(t)$, so does $\lim_{t \to 0} \frac{1}{t^{k-1}} f(t) = 0$.

The compatibility with the wedge product follows from
$$T_t^*(c_1 \wedge c_2) = T_t^* c_1 \wedge T_t^* c_2 \quad \forall c_1, c_2 \in \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)).$$

To prove boundedness, fix a cochain $c \in C^q(\mathfrak{X}(\mathbb{R}^n))$ with $q > 0$ and any $X_1, \ldots, X_q \in \mathfrak{X}(\mathbb{R}^n)$.

Applying the Hadamard lemma to every one of the vector fields X_i shows that there are vector fields $X_k^{(i)} \in \mathfrak{X}(\mathbb{R}^n)$ for all $k = 1, \ldots, q$ and $i = 1, \ldots, n$ so that
$$X_k(x) = X_k(0) + \sum_{i=1}^n x_i \cdot X_k^{(i)}(x), \quad \forall x \in \mathbb{R}^n.$$

Then,
$$(T_t)_* X_k(x) = \frac{1}{t} X_k(0) + \sum_{i=1}^n x_i \cdot X_k^{(i)}(tx) \quad \forall x \in \mathbb{R}^n.$$

Hence, we can rewrite

$$T_t^* c(X_1, \dots, X_q) = c\left(\frac{1}{t} X_1(0) + \sum_{i=1}^n x_i \cdot X_1^{(i)}(tx), \dots, \frac{1}{t} X_q(0) + \sum_{i=1}^n x_i \cdot X_q^{(i)}(tx)\right).$$

Decomposing this expression using multilinearity of c, we find that all the terms whose order in t is lower than $-n$ have to vanish, since any collection of $n+1$ vectors $X_{i_1}(0), \dots, X_{i_{n+1}}(0)$ is linearly dependant and c is skew-symmetric. Note also that on any compact set in $\mathbb{R}^n$, the vector fields $x \mapsto x_i \cdot X_k^{(i)}(tx)$ (and all their derivatives) converge uniformly to the (derivatvies of the) vector field $x \mapsto x_i \cdot X_k^{(i)}(0)$ for $t \to 0$.

Combining the two previous facts, the continuity of c lets us conclude that the limit $\lim_{t\to 0} \frac{1}{t^{-n}} T_t^* c(X_1, \dots, X_q)$ exists. This proves the statement. □

The analysis at the end of the previous proof motivates a different characterization of the filtration:

Lemma 3.6. *A cochain $c \in \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$ is an element in $F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$ if and only if for all polynomial vector fields $X_1, \dots, X_q \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$*

$$\sum_{i=1}^q \deg X_i < k \implies c(X_1, \dots, X_k) = 0.$$

Proof. Note first that for a homogeneous polynomial vector field $X \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ of degree k, we have

$$(T_t)_* X = t^k \cdot X \quad \forall t > 0.$$

Assume first that $c \in F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$. Let $X_1, \dots, X_q \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ be any homogeneous polynomial vector fields with $\sum_{i=1}^q \deg X_i =: r < k$. Then

$$\frac{1}{t^k} \cdot T_t^* c(X_1, \dots, X_q) = t^{r-k} c(X_1, \dots, X_q).$$

Since $r-k < 0$, this can only converge to a finite value as $t \to 0$ if $c(X_1, \dots, X_q) = 0$.

Since all polynomial vector fields are linear combinations of homogeneous ones, this proves the first direction.

On the other hand, assume $c \in \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$ vanishes on all polynomial vector fields whose degree adds to a value smaller k. Let $r := \max\{q+k, 1\}$. Given any vector fields $X_1, \dots, X_q \in \mathfrak{X}(\mathbb{R}^n)$, we may apply the Hadamard lemma to each of them r times to write

$$X_k(x) = Y_k(x) + \sum_{i_1, \dots, i_{r+1}=1}^n x_{i_1} \dots x_{i_{r+1}} Z_k^{(i_1 \dots i_{r+1})}(x) =: Y_k(x) + Z_k(x),$$

where Y_k is a polynomial vector field of degree $\leq r-1$, and $Z_k^{i_1 \dots i_{r+1}} \in \mathfrak{X}(\mathbb{R}^n)$.

Using multilinearity of c, decompose $T_t^* c(X_1, \dots, X_q)$ into summands of the shape $\frac{1}{t^k} T_t^* c$ with all arguments being some Y_k or some Z_k for $k = 1, \dots, q$.

The limits $\lim_{t\to 0} t T_t^* Y_k$ and $\lim_{t\to 0} t^{-r} T_t^* Z_k$ in $\mathfrak{X}(\mathbb{R}^n)$ exist for all $k = 1, \dots, q$. As such, any summand in the decomposition of $\frac{1}{t^k}\left(T_t^* c\right)(X_1, \dots, X_q)$ which contains at least one Z_k as an argument is of the following form with some $s \geq 1$ and $i_1, \dots, i_s, j_1, \dots, j_{q-s} \in \{1, \dots, q\}$:

$$\frac{1}{t^k}\left(T_t^* c\right)\left(Z_{i_1}, \ldots, Z_{i_s}, Y_{j_1}, \ldots, Y_{j_{q-s}}\right)$$
$$= t^{rs-(q-s)-k}(T_t^* c)(t^{-r} \cdot Z_{i_1}, \ldots, t^{-r} \cdot Z_{i_s}, t \cdot Y_{j_1}, \ldots, t \cdot Y_{j_{q-s}}).$$

But since $s \geq 1$ and $r \geq q + k$, we have

$$rs - (q - s) - k \geq s \geq 1,$$

hence, for all these summands, the limit $t \to 0$ exists.

The only summand left to consider is $\frac{1}{t^k} T_t^* c(Y_1, \ldots, Y_q)$. The Y_k are polynomial vector fields, so we may use multilinearity to decompose this term so that we get terms of $\frac{1}{t^k} T_t^* c$ whose arguments are homogeneous polynomial vector fields. In every such summand, $\frac{1}{t^k} T_t^* c$ can be replaced by $t^{\Sigma - k} c$, where Σ is the sum of the degrees of inserted homogeneous vector fields. By assumption on c, every summand where $\Sigma < k$ must vanish.

This implies that as $t \to 0$, the term $\frac{1}{t^k} T_t^* c(Y_1, \ldots, Y_k)$ converges to a finite value. This concludes the proof.

□

Definition 3.7. Given any formal vector field $X \in W_n \cong J_0^\infty \mathfrak{X}(\mathbb{R}^n)$, denote by $\tilde{X}^{(r)} \in \mathfrak{X}(\mathbb{R}^n)$ the polynomial vector field of degree r corresponding to the jet of X at zero.

Define for all $k \in \mathbb{Z}$ and $q \geq 1$ the maps

$$\gamma_k : F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n)) \to C^q_{(k)}(W_n), \quad \beta_k : C^q_{(k)}(W_n) \to F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n)),$$
$$(\gamma_k c)(X_1, \ldots, X_q) := \lim_{r\to\infty} \lim_{t\to 0} t^{-k} (T_t^* c)(\tilde{X}_1^{(r)}, \ldots, \tilde{X}_q^{(r)}),$$
$$(\beta_k c)(Y_1, \ldots, Y_k) := c\left(j_0^\infty Y_1, \ldots, j_0^\infty Y_k\right),$$

for all $X_1, \ldots, X_q \in W_n$ and $Y_1, \ldots, Y_q \in \mathfrak{X}(\mathbb{R}^n)$.

Lemma 3.8. *The maps β_k and γ_k are well-defined chain maps with $\gamma_k \circ \beta_k = \mathrm{id}$.*

Proof. Note first that γ_k is well-defined: The limit $t \to 0$ exists on the domain $F^k \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$, and the sequence is is eventually constant in r, since the cochain $\lim_{t\to 0} T_t^* c$ vanishes on homogeneous vector fields with sufficiently high degree. Further, if $c \in F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$, then by Lemma 3.6, it vanishes on polynomial vector fields whose sum of degree is smaller than k. It also vanishes on homogeneous arguments whose sum of degrees is larger than k in the limit $t \to 0$. Hence $\gamma_k c \in C^q_{(k)}(W_n)$.

Analogously, if $c \in C^k_{(r)}(W_n)$, then $\beta_k c$ vanishes on polynomial vector fields whose sum of degrees is smaller than k, hence Lemma 3.6 implies $\beta_k c \in F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$.

Recall that $P_k \subset \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ denotes the polynomial vector fields of degree k. The identification of a finite formal vector field X_n with its Taylor polynomial in $\mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ is a Lie algebra morphism, and so is the pushforward of a vector field by the diffeomorphism T_t. Hence γ_k is a chain map.

On the other hand, β_k is a chain map since taking the infinite jet of a vector field at zero is a Lie algebra morphism $\mathfrak{X}(\mathbb{R}^n) \to W_n$.

Consider now for any homogeneous formal vector fields $X_1, \dots, X_n \in W_n$ and $c \in C^q_{(k)}(W_n)$ the expression:

$$\begin{aligned}(\gamma_k \beta_k c)(X_1, \dots, X_q) &= \lim_{r\to\infty} \lim_{t\to 0} t^{-k}(T_t^* \beta_k c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}) \\ &= \lim_{r\to\infty} \lim_{t\to 0} t^{\left(\sum_{i=1}^q \deg \tilde{X}_i^{(r)}\right)-k}(\beta_k c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}).\end{aligned}$$

If $\sum_{i=1}^q \deg \tilde{X}_i^{(r)} < k$, then $(\beta_k c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}) = 0$ by the characterization of the filtration from Lemma 3.6 and since $\beta_k c \in F^k \tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$.

If $\sum_{i=1}^q \deg \tilde{X}_i^{(r)} > k$ for any $r \in \mathbb{N}$, then also for all $r' \geq r$ and

$$\lim_{t\to 0} t^{\left(\sum_{i=1}^q \deg X_i\right)-k}(\beta_k c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}) = 0 \quad \forall r' \geq r.$$

Finally, if $\sum_{i=1}^q \deg \tilde{X}_i^{(r)} = k$ for almost all $r \in \mathbb{N}$, then

$$\begin{aligned}&\lim_{t\to 0} t^{\left(\sum_{i=1}^q \deg \tilde{X}_i^{(r)}\right)-k}(\beta_k c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}) \\ &\qquad = (\beta_k c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}) = c(X_1, \dots, X_q)\end{aligned}$$

for almost all r, and hence $\gamma_k \beta_k c = c$. This concludes the proof. □

Lemma 3.9. *For every $k \in \mathbb{Z}$, we have the short exact sequence of cochain complexes*

$$0 \to F^{k+1}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^k \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \xrightarrow{\gamma_k} \tilde{C}^\bullet_{(k)}(W_n) \to 0.$$

Proof. The inclusion $F^{k+1}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^k\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is a chain map because $\{F^k\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))\}_{k\in\mathbb{Z}}$ makes up a filtration of chain complexes. On the other hand, γ_k is a chain map due to Lemma 3.8. Hence, the sequence is a sequence of chain complexes.

The injectivity of the first map is clear, and the second map is surjective, since it is split by β_k. Exactness at the middle term follows from the characterization of the filtration in Lemma 3.6. This concludes the proof. □

Lemma 3.10. *Define for all $t > 0$ the map $H_t : \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ via*

$$H_t := t^{-1} T_t^* \circ (E \lrcorner \cdot),$$

where $E \in \mathfrak{X}(\mathbb{R}^n)$ is the Euler vector field, with $E(x) = \sum_{i=1}^n x_i \partial_i$.

Then, we have for the scalings of vector fields and cochains, respectively:

$$\frac{d}{dt}(T_t)_* = t^{-1}(T_t)_* \circ \mathcal{L}_E, \quad \frac{d}{dt}T_t^* = H_t d + dH_t,$$

denoting by $\mathcal{L}_E$ the usual Lie derivative with respect to E.

Proof. Note first that for all vector fields $X = \sum_{i=1}^n f_i \partial_i \in \mathfrak{X}(\mathbb{R}^n)$ and for all vectors $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ we have

$$\left(\frac{d}{dt}(T_t)_* X\right)(p) := -\frac{1}{t^2}X(tp) + \sum_{i,j=1}^n \frac{p_j}{t}(\partial_j f_i)(tp)\partial_i.$$

But for the Euler vector field $E \in \mathfrak{X}(\mathbb{R}^n)$ it holds that

$$(\mathcal{L}_E X)(p) = [E, X](p) = -X(p) + \sum_{i,j=1}^n p_j(\partial_j f_i)(p)\partial_i).$$

Thus, for the contraction of vector fields we get the first desired formula

$$\frac{d}{dt}(T_t)_* = \mathcal{L}_E \circ (t^{-1}(T_t)_*) = (t^{-1}(T_t)_*) \circ \mathcal{L}_E.$$

The function $t \mapsto (T_t^* c)(X_1, \dots, X_k)$ is smooth in t for all $X_i \in \Gamma(T\mathbb{R}^n)$, so

$$\frac{d}{dt}(T_t^* c)(X_1, \dots, X_k) = \sum_{i=1}^{k} c\left((T_t)_* X_1, \dots, \frac{d}{dt}(T_t)_* X_i, \dots, (T_t)_* X_k\right).$$

The Lie derivative $\mathcal{L}_Y$ on forms $\omega \in \Omega^k(M)$ fulfils the well-known magic Cartan formula

$$\mathcal{L}_Y \omega = Y \lrcorner\, d_{\mathrm{dR}} \omega + d_{\mathrm{dR}}(Y \lrcorner\, \omega).$$

If we replace forms ω with cochains $c \in \tilde{C}^k(\mathfrak{X}(\mathbb{R}^n))$ and the Lie derivative of forms for $Y \in \mathfrak{X}(\mathbb{R}^n)$ with

$$\mathcal{L}_Y : \tilde{C}^k(\mathfrak{X}(\mathbb{R}^n)) \to \tilde{C}^k(\mathfrak{X}(\mathbb{R}^n)),$$

$$(\mathcal{L}_Y c)(X_1, \dots, X_k) := \sum_{i=1}^{k} c(X_1, \dots, \mathcal{L}_Y X_i, \dots, X_k),$$

then the analogous magic formula holds for this Lie derivative, with the de Rham differential replaced by the Lie algebra differential.

Thus, the second formula follows:

$$\frac{d}{dt} T_t^* c = (t^{-1}(T_t)_*) \circ \mathcal{L}_E c = (t^{-t}(T_t)_*) \circ (E \lrcorner\, dc + d(E \lrcorner\, c)) = H_t dc + dH_t c.$$

□

Corollary 3.11. The complex $F^1\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is acyclic.

Proof. Note that if $c \in F^1\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$, then $\lim_{t\to 0}(T_t^* c)(X_1, \dots, X_n) = 0$ for all $X_1, \dots, X_n \in \mathfrak{X}(\mathbb{R}^n)$. But then

$$\begin{aligned} c(X_1, \dots, X_k) &= (T_1^* c)(X_1, \dots, X_k) - \lim_{t\to 0}(T_t^* c)(X_1, \dots, X_k) \\ &= \int_0^1 \frac{d}{dt}(T_t^* c)(X_1, \dots, X_k) dt \\ &= \int_0^1 \left(\frac{d}{dt} T_t^* c\right)(X_1, \dots, X_k) dt \\ &= \int_0^1 (H_t dc + dH_t c)(X_1, \dots, X_k) dt =: (Kdc + dKc)(X_1, \dots, X_k), \end{aligned}$$

where we defined $K := \int_0^1 H_t dt$. Hence K is a chain homotopy between the identity map and zero on $F^1\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$, which proves the statement. □

Finally we can state a variation of Lemma 1 in Section 2.4.B. of [9], proven with the lengthier method from [10].

Theorem 3.12. *The inclusion*

$$F^0\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \hookrightarrow C^\bullet(\mathfrak{X}(\mathbb{R}^n))$$

and the maps

$$\gamma_0 : F^0\tilde{C}^q(\mathfrak{X}(\mathbb{R}^n)) \to C^q_{(0)}(W_n), \quad \beta_0 : C^q_{(0)}(W_n) \to F^0\tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$$

from Definition 3.7 are quasi-isomorphisms and algebra morphisms with respect to the wedge product of cochains.

In particular,

$$H^\bullet(\mathfrak{X}(\mathbb{R}^n)) \cong H^\bullet(W_n),$$

and the wedge product of two cohomology classes of nonzero degree in $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is trivial.

Proof. The compatibility of the maps with the wedge product is immediate from the multiplicativity of the filtration $F^k\tilde{C}^q(\mathfrak{X}(\mathbb{R}^n))$ and the formulas for γ_0, β_0. By Lemma 3.9, for every $k \in \mathbb{Z}$ there is an exact sequence

$$0 \to F^{k+1}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^k\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \stackrel{\gamma_k}{\to} \tilde{C}^\bullet_{(k)}(W_n) \to 0. \tag{3.1}$$

Insert $k = 0$ here. The arising long exact sequence in cohomology and the acyclicity of $F^1\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ by Corollary 3.11 imply

$$H^\bullet(F^0\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))) \cong \tilde{H}^\bullet_{(0)}(W_n) \stackrel{\text{Prop. 2.14}}{\cong} \tilde{H}^\bullet(W_n).$$

On the other hand, let $k = -1, \dots, -n$ in (3.1). By Proposition 2.14, the complexes $\tilde{C}^\bullet_{(k)}(W_n)$ are acyclic and

$$\begin{aligned} H^\bullet(F^0\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))) &\cong H^\bullet(F^{-1}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))) \\ &\cong \dots \\ &\cong H^\bullet(F^{-n}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))) \\ &\stackrel{\text{Lem. 3.5}}{=} H^\bullet(\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))) = \tilde{H}^\bullet(\mathfrak{X}(\mathbb{R}^n)). \end{aligned}$$

Combining the last two calculations concludes the proof. □

Remark 3.13. While we do require the above, elaborate groundwork on the structure of $C^\bullet(\mathfrak{X}(\mathbb{R}^n))$, the above result is proven in a considerably shorter way in [9, Section 2.4.B, Lemma 1]. For the sake of completion, we want to sketch the idea and highlight what a reader of the source material needs to be mindful of:

By Taylor expansion, one can decompose $\mathfrak{X}(\mathbb{R}^n) = P_k \oplus V_k$, where P_i are the polynomial vector fields of degree k and V_k are those vector fields which vanish in zero up to order k. Dually one can thus get the following decomposition (by basically taking $k \to \infty$):

$$C^\bullet(\mathfrak{X}(\mathbb{R}^n)) = C^\bullet(W_n) \oplus B^\bullet,$$

where

$$B^\bullet := \{c \in C^\bullet(\mathfrak{X}(\mathbb{R}^n)) : \lim_{t\to 0} t^k T_t c = 0 \quad \forall k \in \mathbb{Z}\}.$$

In [9], it is claimed that elements $c \in B^\bullet$ are characterized by the property $\lim_{t\to 0} T_t^* c = 0$. All cochains in B^k fulfil this property, but it is not an equivalence. However, by the same homotopy formula as in Corollary 3.11, we can conclude that $B^\bullet$ is an acyclic subcomplex. Hence $H^\bullet(\mathfrak{X}(\mathbb{R}^n)) \cong H^\bullet(W_n)$.

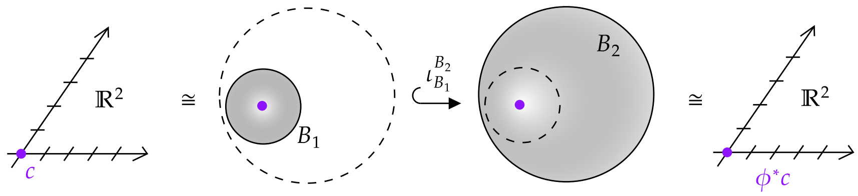

FIGURE 3. A visualization of the extension process in the proof of Proposition 3.16, for the example $n = 2$. The colored dot represents the support of a cochain $c \in H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ from the image of the quasi-isomorphism map β, and how the support is transformed by the map ϕ from (3.2). As the support of c equals a single point, the resulting cochain $\phi^* c$ only depends on the transformation behaviour of ϕ around this point.

3.2. **Transformation of local Gelfand-Fuks cohomology.** The previous subsection concludes the analysis of the cohomology of $\mathfrak{X}(\mathbb{R}^n)$. However, this local cohomology constitues an important building block to understand the Gelfand-Fuks cohomology for arbitrary smooth manifolds M. As such, we will explore some properties related to *extension* of the cochains from a smaller to a larger open set of M. The ideas here are also outlined in [10].

Definition 3.14. Let M be a smooth manifold, and $U \subset V$ open subsets of M.

i) Define the *extension of cochains*

$$\iota_U^V : \tilde{C}^\bullet(\mathfrak{X}(U)) \to \tilde{C}^\bullet(\mathfrak{X}(V)),$$
$$(\iota_U^V c)(X_1, \ldots, X_k) := c(X_1\big|_U, \ldots, X_k\big|_U)$$

for all $c \in \tilde{C}^k(\mathfrak{X}(U))$, $X_1, \ldots, X_k \in \mathfrak{X}(V)$.

ii) The extension of cochains induces an *extension of cohomology classes*

$$\iota_U^V : \tilde{H}^\bullet(\mathfrak{X}(U)) \to \tilde{H}^\bullet(\mathfrak{X}(V))$$

which we will denote with the same symbol ι_U^V by an abuse of notation.

These extension maps are transitive with respect to inclusions $U \subset V \subset W$, which proves the following:

Lemma 3.15. *The structure maps* $\{\iota_U^V\}$ *make the assignment* $U \mapsto \tilde{C}^\bullet(\mathfrak{X}(U))$ *into a precosheaf.*

While sheaf theory is well known, the dual concept of cosheaves is less commonly considered. Hence, for self-containedness of this document, we direct the reader to Appendix A, or [18] for a more detailled study of both sheaf and cosheaf theory. We will delve deeper into the cosheaf-theoretic aspects of Gelfand-Fuks cochains in Section 4.

Proposition 3.16. Given two open balls $B_1 \subset B_2 \subset \mathbb{R}^n$, the extension map $\iota_{B_1}^{B_2} : \tilde{H}^\bullet(\mathfrak{X}(B_1)) \to \tilde{H}^\bullet(\mathfrak{X}(B_2))$ is an algebra isomorphism with respect to the wedge product of cohomology classes.

Proof. Fix diffeomorphisms $B_1 \cong \mathbb{R}^n, B_2 \cong \mathbb{R}^n$. Now, consider the composition

$$\phi : \mathbb{R}^n \to B_1 \to B_2 \to \mathbb{R}^n. \tag{3.2}$$

Without loss of generality, assume that ϕ fixes zero. Recall the quasi-isomorphism $\beta_0 : \tilde{C}^\bullet(W_n) \to \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ from Definition 3.7. Every cohomology class in $H^\bullet(\mathfrak{X}(B_1))$ has a representative of the form $\beta_0 c$, and the pullback of ϕ acts on cochains $\beta_0 c$ by pullback of the infinity-jet of a local diffeomorphism at zero, see also Lemma 2.2).

This is clearly invertible and hence shows that the induced map

$$\tilde{H}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to \tilde{H}^\bullet(\mathfrak{X}(B_1)) \to \tilde{H}^\bullet(\mathfrak{X}(B_2)) \to \tilde{H}^\bullet(\mathfrak{X}(\mathbb{R}^n))$$

is an isomorphism. But since the first and last arrow in the above composition are both isomorphisms, so is the one in the middle.

The compatibility of the wedge product is a straightforward calculation on the level of cochains. □

Proposition 3.17. Let $M := \bigsqcup_{i=1}^r \mathbb{R}^n$ be a disjoint collection of copies of $\mathbb{R}^n$. Then every choice of order on the copies of $\mathbb{R}^n$ induces an algebra isomorphism

$$\tilde{H}^\bullet\left(\mathfrak{X}\left(M\right)\right) \cong \left(H^\bullet(\mathfrak{X}(\mathbb{R}^n))^{\otimes^r}\right)_{\text{red}}.$$

Here, recall that the index "red" denotes the reduced cohomology.

Proof. We mimic the proof for the $r = 1$ situation, but we expand the scaling of $\mathbb{R}^n$ to the same scaling in every copy of $\mathbb{R}^n$:

$$T_t : \bigsqcup_{i=1}^r \mathbb{R}^n \to \bigsqcup_{i=1}^r \mathbb{R}^n, \quad x \mapsto \frac{x}{t}.$$

The definition of the corresponding spaces $F^k\tilde{C}^q(\mathfrak{X}(M))$ is identical to in the $r = 1$ case, and by the same proofs, they constitute a descended, filtration that is bounded from below with

$$F^k\tilde{C}^\bullet(\mathfrak{X}(M)) = \tilde{C}^\bullet(\mathfrak{X}(M)) \quad \forall k \leq -r \cdot n.$$

In analogy to Definition 3.7, we can define a map $\gamma_k^{(r)}$, which, together with some choice of ordering on the components of M gives rise to an exact sequence for every $k \in \mathbb{Z}$:

$$0 \to F^{k+1}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^k\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \xrightarrow{\gamma_k^{(r)}} \left(\bigoplus_{k_1+\cdots+k_r=k} C^\bullet_{(k_1)}(W_n) \otimes \cdots \otimes C^\bullet_{(k_r)}(W_n)\right)_{\text{red}} \to 0.$$

For the tensor product complex on the right hand side, we can use the Künneth theorem to calculate its cohomology, and due to acyclicity of $C^\bullet_{(k)}(W_n)$ for $k \neq 0$, the only one of the complexes with nontrivial cohomology is the one with the condition $k_1 + \cdots + k_r = 0$. By the same steps as in Corollary 3.11 and Theorem 3.12 we arrive at the desired isomorphism of vector spaces.

This isomorphism respects the wedge product, as we see with the arising quasi-isomorphism

$$\beta_0^{(r)} : \left(C^\bullet_{(0)}(W_n)^{\otimes^r}\right)_{\text{red}} \to \tilde{C}^\bullet(\mathfrak{X}(M)),$$
$$c_1 \otimes \cdots \otimes c_r \mapsto \beta_0^{(1),1} c_1 \wedge \cdots \wedge \beta_0^{(1),r} c_r,$$

where $\beta_0^{(1),k}$ maps formal cochains exactly like the map β_0 from the Definition 3.7, but all jets of vector fields are evaluated at the zero in the k-th copy of $\mathbb{R}^n$. Because the β_0 in the $r = 1$ case respect the wedge product, so does $\beta_0^{(r)}$. □

The formula for the quasi-isomorphism $\beta_0^{(r)}$ from the previous proof implies:

Corollary 3.18. Let $B_1, B_2 \subset \mathbb{R}^n$ be two disjoint open balls, whose union is contained in another open ball $C \subset \mathbb{R}^n$. Then the extension map

$$\iota_{B_1 \cup B_2}^C : (H^\bullet(\mathfrak{X}(B_1)) \otimes H^\bullet(\mathfrak{X}(B_2)))_{\text{red}} \cong \tilde{H}^\bullet(\mathfrak{X}(B_1 \cup B_2)) \to \tilde{H}^\bullet(\mathfrak{X}(C))$$

is given by

$$[c_1] \otimes [c_2] \mapsto [\iota_{B_i}^C c_1 \wedge \iota_{B_2}^C c_2].$$

We also want to make a quick remark on how *translation* on the Euclidean space acts on cohomology:

Lemma 3.19. *For all $a \in \mathbb{R}^n$, given the diffeomorphism*

$$\tau_a : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x + a.$$

This induces a map $(\tau_a)_*$ on vector fields $\mathfrak{X}(\mathbb{R}^n)$ and a map τ_a^* cochains $\tilde{C}^\bullet(\mathbb{R}^n)$ by pullback, fulfilling the identity

$$\tau_a^* - \text{id} = dK + Kd,$$

where $K := -\sum_{i=1}^n \int_0^1 \tau_{ta}^*(\partial_i \lrcorner \cdot)dt$.

Proof. For all $a, x \in \mathbb{R}^n$ and $X \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\begin{aligned}(\tau_a)_* X(x) - X(x) &= X(x-a) - X(x) \\ &= -\sum_{i=1}^n \int_0^1 (\partial_i X)(x - ta)\, dt \\ &= -\sum_{i=1}^n \int_0^1 \tau_{ta}^*[\partial_i, X]\, dt = -\sum_{i=1}^n \int_0^1 \tau_{ta}^* \mathcal{L}_{\partial_i} X\, dt.\end{aligned}$$

Hence, by the same methods as in the proof for Corollary 3.11,

$$\tau_a^* c - c = -\sum_{i=1}^n \int_0^1 \tau_{ta}^* \mathcal{L}_{\partial_i} c\, dt = -\sum_{i=1}^n \int_0^1 d(\tau_{ta}^*(\partial_i \lrcorner c)) + \tau_{ta}^*(\partial_i \lrcorner dc)\, dt.$$

This concludes the proof. □

3.3. **A cosheaf of local Gelfand-Fuks cohomology.** We conclude this section by studying an important precosheaf on a base (see Appendix A.4) arising from the study of local Gelfand-Fuks cohomology. The assignment $U \mapsto H^\bullet(\mathfrak{X}(U))$ is not a cosheaf on M, since globally, Gelfand-Fuks cohomology is controlled by topological effects which are not present in the local case. However, we can nonetheless extend the assignment $U \mapsto H^\bullet(\mathfrak{X}(U))$ on open balls U to a unique cosheaf, which will become useful later.

Definition 3.20. Let M be a smooth manifold. Consider the topological basis of M given by

$$\mathcal{B}_{\mathbb{R}^n} := \{U \subset M : U \text{ diffeomorphic to } \mathbb{R}^n\}.$$

We define, for all $q \geq 0$, a precosheaf $\mathcal{H}^q$ on the base $\mathcal{B}_{\mathbb{R}^n}$ by associating to $U \subset M$ the set $H^q(\mathfrak{X}(U))$, and defining the extension maps as follows:

- For $q = 0$, set for all $U, V \in \mathcal{B}_{\mathbb{R}^n}$ with $U \subset V$ the map $\iota_U^V : H^0(\mathfrak{X}(U)) \to H^0(\mathfrak{X}(V))$ to be equal to the identity $\mathbb{R} \to \mathbb{R}$.
- For $q > 0$, define

$$\iota_U^V : H^q(\mathfrak{X}(U)) \to H^q(\mathfrak{X}(V))$$

  to be simply the extension of cohomology classes, see Definition 3.14.

Proposition 3.21. Let M be a smooth manifold, then the precosheaves $\mathcal{H}^q$ on the base $\mathcal{B}_{\mathbb{R}^n}$ are cosheaves on this base. As such, they extend to unique cosheaves $\mathcal{H}^q$ on M.

Proof. By Proposition 3.16 we know that all extension maps of the precosheaf on the base $\mathcal{B}_{\mathbb{R}^n}$ are isomorphisms. From this both cosheaf properties follow straightforwardly. □

Proposition 3.22. If M is orientable, then the cosheaf $\mathcal{H}^\bullet$ is isomorphic to the constant cosheaf $U \mapsto H^\bullet(W_n)$ on M.

Proof. Choose a smooth, oriented atlas $\mathcal{A}$ for M. By refinement, we may assume that all open sets in the atlas are diffeomorphic to open balls in $\mathbb{R}^n$ and that they constitute a topological base $\mathcal{B}$ of M. Hence we can define a precosheaf on this base via the assignment $U \mapsto H^\bullet(\mathfrak{X}(U))$ and the same extension maps as in Definition 3.20. This is a cosheaf on this base by the same argument as in Proposition 3.21, and since $\mathcal{B} \subset \mathcal{B}_{\mathbb{R}^n}$, this cosheaf on this base extends to the cosheaf $\mathcal{H}$ on M defined in Definition 3.20.

Let (U, ϕ) and (V, ψ) be two charts in $\mathcal{A}$, and assume $U \subset V$. Recall the quasi-isomorphisms β_0, γ_0 from Definition 3.7, and denote by ϕ^*, ψ^* the pullback of cochains arising from ϕ and ψ.

Consider the map

$$\Xi := \gamma_0 \circ \psi^* \circ \iota_U^V \circ (\phi^{-1})^* \circ \beta_0 : C^\bullet(W_n) \to C^\bullet(W_n).$$

By unravelling the definition of Ξ, we see that it equals the pullback of $C^\bullet(W_n)$ by the local diffeomorphism $\psi\big|_U \circ \phi^{-1}$ (compare Lemma 2.2).

In Lemma 3.19, we have shown that if $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ is the translation map for $a \in \mathbb{R}^n$, then its pullback on cochains induces the identity map on cohomology. Hence, as maps on cohomology, $[\phi] = [\tau_a \circ \phi]$ for all $a \in \mathbb{R}^n$, so we may assume that $\psi\big|_U \circ \phi^{-1}$ fixes zero.

In this case, Ξ acts on $H^\bullet(W_n)$ by an element of $J_0^\infty \operatorname{Diff}(\mathbb{R}^n)$, the group of ∞-jets of diffeomorphisms on $\mathbb{R}^n$ that fix zero.

Now, since $\mathcal{A}$ is oriented, the transition function $\psi\big|_U \circ \phi^{-1}$ is a positively oriented diffeomorphism of $\mathbb{R}^n$. As such, the corresponding jets lie in the identity component of the Lie groups $J_0^r \operatorname{Diff}(\mathbb{R}^n)$ for $r \in \mathbb{N}$. The (Fréchet) Lie algebra of $J_0^\infty \operatorname{Diff}(\mathbb{R}^n)$ is equal to the projective limit

$$J_0^\infty \mathfrak{X}(M) \cong \widehat{\bigoplus_{k \geq 0}} \mathfrak{g}_k \subset W_n,$$

where the $\mathfrak{g}_k$ are the homogeneous graded components of W_n (see also [19, Chapter IV.13]), and the action of $J_0^\infty TM \subset W_n$ on W_n is canonical action of a Lie subalgebra on its ambient Lie algebra. Hence the Lie algebras $J_0^r\mathfrak{X}(M)$ of the finite dimensional, nilpotent Lie groups $J_0^r \operatorname{Diff}(\mathbb{R}^n)$ are quotients of W_n.

Fix an equivalence class $[c] \in H^\bullet(W_n)$ with representative c. Since c is only nonzero on a finite-dimensional subspace of W_n, we have

$$\Xi(c) = \Xi_r(c)$$

for some finite jet $\Xi_r \in J_0^r \operatorname{Diff}(\mathbb{R}^n)$.

The exponential from a nilpotent Lie algebra is surjective onto the identity component of the corresponding Lie group, hence there is some r-jet $X_r \in J_0^r TM$ and some $X \in J_0^\infty TM \subset W_n$ extending X_r so that

$$\Xi(c) = \Xi_r(c) = \exp([X]_r) \cdot c = \exp(X) \cdot c.$$

The action of a Lie algebra on its cohomology is trivial by Corollary 2.12, and as a consequence $\Xi([c]) = [c]$. Since $[c] \in H^\bullet(W_n)$ was arbitrary, we have

$$[\gamma_0 \circ \psi^* \circ \iota_U^V \circ (\phi^{-1})^* \circ \beta_0] = [\Xi] = \mathrm{id}\big|_{H^\bullet(W_n)}.$$

This implies that we can define a cosheaf isomorphism between the constant cosheaf $U \mapsto H^\bullet(W_n)$ and $\mathcal{H}$ on the base $\mathcal{B}$, via

$$H^\bullet(W_n) \underset{\gamma_0\circ(\psi^{-1})^*}{\overset{\phi^*\circ\beta_0}{\rightleftarrows}} \mathcal{H}.$$

By Theorem A.11 this extends to an isomorphism of the constant cosheaf on M and $\mathcal{H}$, and the statement is proven. □

Remark 3.23. In the non-orientable case, the previous result implies that the cosheaf $\mathcal{H}$ is *locally constant*, i.e. for every point $x \in M$ there is an open neighbourhood U of x such that $\mathcal{H}\big|_U$ is a constant cosheaf on U.

4. Gelfand-Fuks cohomology for smooth manifolds

In this section, we construct a spectral sequence due to Gelfand and Fuks that calculates the continuous Lie algebra cohomology for smooth manifolds, following a local-to-global principle using sheaf theoretic ideas.

The spectral sequence itself was constructed in [1], by a complicated global analysis of the cochain spaces $C^\bullet(\mathfrak{X}(M))$ in terms of explicit distributions.

The proposed local-to-global principle has originally been outlined in [10] and [20], and, according to the last reference, was initially suggested by Segal.

In these latter two references, there are some subtleties that remained unaddressed: They indirectly claim that the assignment of open sets U to $C^\bullet(\mathfrak{X}(U))$ is an *cosheaf* of graded vector spaces, i.e. its Čech homology (see Appendix A) vanishes with respect to every good cover $\mathcal{U}$ of M.

This is not true, see Example 4.1. We present a proof that works around this problem by using so-called *k-good* covers, an adaptation to the concept of a good cover originating from [12]. This was inspired by the recent preprint [13], treating Gelfand-Fuks cohomology in the setting of factorization algebras. To the knowledge of the author, this proof does not show up in the literature in the current form.

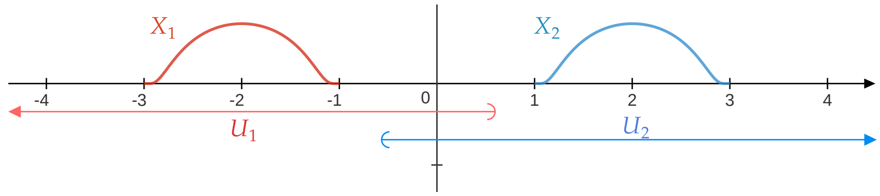

FIGURE 4. A possible visualization of Example 4.1 for $M = \mathbb{R}$, where we identify vector fields $\mathfrak{X}(\mathbb{R}) \cong C^\infty(\mathbb{R})$. An example for a cochain c which is nonzero on these vector fields would be $c(f_1\partial, f_2\partial) = f_1(2)f_2(-2) - f_1(-2)f_2(2)$.

However, we want to emphasize that this subtlety does not influence the validity of the final results from Bott and Segal. The mistake is not repeated in [4] and [9], where similar, but more sophisticated Čech-theoretic methods are used.

Regardless, our proof gives a more elementary way to calculate the Gelfand-Fuks spectral sequence for *k-diagonal cohomology*, an approximation of Gelfand-Fuks cohomology which we will introduce in the following section. The expression for the spectral sequence has been given in [9] without an explicit proof for $k \neq 1$; the proof in [10] is only a sketch, with previously mentioned issues, and the proof in [1] is substantially more involved.

4.1. **Diagonal Filtration.** Fix a smooth manifold M of dimension n.

The previously established precosheaf structure (see Definition 3.14) of the cochains $C^\bullet(\mathfrak{X}(M))$ does *not* extend to a cosheaf structure.

Example 4.1. Let M be a smooth manifold of nonzero dimension. Then there are smooth, nonzero vector fields $X_1, X_2 \in \mathfrak{X}(M)$ with $\operatorname{supp} X_1 \cap \operatorname{supp} X_2 = \emptyset$, some cochain $c \in C^2(\mathfrak{X}(M))$ with $c(X_1, X_2) \neq 0$, and some open cover $\{U_1, U_2\}$ of M with

$$\operatorname{supp} X_1 \cap U_2 = \emptyset, \quad \operatorname{supp} X_2 \cap U_1 = \emptyset.$$

If the assignment of an open set $U \subset M$ to $C^2(\mathfrak{X}(U))$ was a cosheaf, then there would exist $c_i \in C^\bullet(\mathfrak{X}(U_i))$ for $i = 1, 2$ with

$$c = \iota^M_{U_1} c_1 + \iota^M_{U_2} c_2.$$

But then, because $X_2\big|_{U_1} = X_1\big|_{U_2} = 0$, it follows that

$$0 \neq c(X_1, X_2) = \left(\iota^M_{U_1} c_1 + \iota^M_{U_2} c_2\right)(X_1, X_2) = 0.$$

A clear contradiction, hence, the precosheaf $U \mapsto C^\bullet(\mathfrak{X}(U))$ is not a cosheaf.

Hence, as we increase the number of arguments in our cochains, we may get *locality* or *diagonality problems* as in the above proof. It will be valuable to replace these spaces by certain diagonal replacements:

Definition 4.2. Let U be an open subset of M, and $q > 0$.

i) Define the graded vector space $B^\bullet(\mathfrak{X}(U)) := \bigoplus_{k \geq 1} B^q(\mathfrak{X}(U))$, where

$$B^q(\mathfrak{X}(U)) := \{c : \mathfrak{X}(U)^q \to \mathbb{R} \mid c \text{ multilinear and jointly continuous}\}.$$

ii) Given a collection of vector fields $\{X_1, \dots, X_q\} \subset \mathfrak{X}(U)$, we say that this collection *has the property* Δ_k if for every finite set $\Gamma \subset U$ of k arbitrary points, there is an X_i that vanishes in a neighbourhood of Γ.
iii) We define the *k-diagonal distributions* as those $c \in B^q(\mathfrak{X}(U))$ with

$$\{X_1, \dots, X_q\} \text{ has property } \Delta_k \implies c(X_1, \dots, X_q) = 0.$$

Their collection is denoted $\Delta_k B^q(\mathfrak{X}(U))$.
iv) Define the *k-diagonal cochains* $\Delta_k C^q(\mathfrak{X}(U)) \subset C^q(\mathfrak{X}(M))$ as the skew-symmetric cochains which are contained in $\Delta_k B^q(\mathfrak{X}(U))$.

Proposition 4.3. For all $k \geq 1$ and all open $U \subset M$, we have the ascending chain

$$\begin{aligned} 0 =: \Delta_0 C^k(\mathfrak{X}(U)) \subset \Delta_1 C^n(\mathfrak{X}(U)) \subset \dots \\ \subset \Delta_{k-1} C^k(\mathfrak{X}(U)) \subset \Delta_k C^k(\mathfrak{X}(U)) = C^k(\mathfrak{X}(U)). \end{aligned}$$

Further, the $\Delta_k C^\bullet(\mathfrak{X}(U))$ constitute a multiplicative filtration of the chain complex $C^\bullet(\mathfrak{X}(U))$.

Proof. The chain follows directly from the definition, since if a set $\{X_1, \dots, X_q\}$ of vector fields has the property Δ_k, it also has the property Δ_{k-1}.

Further, a set $\{X_1, \dots, X_k\}$ of k vector fields can only have the property Δ_k if one of the X_i zero everywhere. Hence $\Delta_k C^k(\mathfrak{X}(M)) = C^k(\mathfrak{X}(M))$. This shows the first part of the proposition.

Further, notice that if $\{X_1, \dots, X_{q+1}\}$ has the property Δ_k, so does the collection $\{[X_1, X_2], X_3, \dots, X_q\}$. From this it follows that

$$d(\Delta_k C^q(\mathfrak{X}(M)) \subseteq \Delta_k C^{q+1}\mathfrak{X}(M)).$$

Lastly, if $\{X_1, \dots, X_{q+r}\}$ has the property Δ_{k+l}, then $\{X_1, \dots, X_q\}$ has the property Δ_k or $\{X_{q+1}, \dots, X_{q+r}\}$ has the property Δ_l. Hence

$$\Delta_k C^\bullet(\mathfrak{X}(M)) \wedge \Delta_l C^\bullet(\mathfrak{X}(M)) \subseteq \Delta_{k+l} C^\bullet(\mathfrak{X}(M)).$$

□

Example 4.4. A set $\{X_1, \dots, X_k\} \subset \mathfrak{X}(M)$ has the property Δ_1 if and only if

$$\bigcap_{i=1}^{k} \operatorname{supp} X_i = \emptyset.$$

Hence, $\Delta_1 C^\bullet(\mathfrak{X}(M))$ consists of the cochains which vanish when the inserted vector fields have disjoint support. These are also called the *diagonal cochains.*

Note that $\Delta_q C^q(\mathfrak{X}(U)) = C^q(\mathfrak{X}(U))$, hence

$$\Delta_k H^q(\mathfrak{X}(U)) = H^q(\mathfrak{X}(U)) \quad \forall k \geq q+1.$$

To put this in terms of more sheaflike data, let us view these cochains through a different lens.

Definition 4.5. Given $q \geq 1$ and the canonical projections $\mathsf{pr}_1, \dots, \mathsf{pr}_q : M^q \to M$, consider the vector bundle

$$\boxtimes^q TM := \bigotimes_{i=1}^{q} \mathsf{pr}_i^* TM \to M^q.$$

Equipping the space of sections $\mathfrak{X}(M)$ with its standard Fréchet topology, the Schwartz kernel theorem for smooth manifolds (see for example [21], or [22, Chapter 51] for the statement for trivial vector bundles) tells us that there is a natural vector space isomorphism

$$B^q(\mathfrak{X}(M)) \cong \Gamma(\boxtimes^q TM)^*, \tag{4.1}$$

the star denoting the continuous dual with respect to the Fréchet topology.

This isomorphism is dual to the map

$$\begin{aligned} \mathfrak{X}(M) \otimes \cdots \otimes \mathfrak{X}(M) \to \Gamma(\boxtimes^q TM), \quad (X_1, \ldots, X_q) \mapsto X_1 \boxtimes \cdots \boxtimes X_q, \\ (X_1 \boxtimes \cdots \boxtimes X_q)(x_1, \ldots, x_q) := X_1(x_1) \otimes \cdots \otimes X_q(x_q) \quad \forall x_1, \ldots, x_q \in M. \end{aligned} \tag{4.2}$$

Definition 4.6. Given $k \in \mathbb{N}$, let

$$M^q_k := \{(x_1, \ldots, x_q) \in M^q : |\{x_{i_1}, \ldots, x_{i_{k+1}}\}| \leq k\}.$$

In other words, $M^q_k \subset M^q$ is the set of $(x_1, \ldots, x_q) \in M^q$ so that there exists no subset $\{x_{i_1}, \ldots, x_{i_{k+1}}\} \subset \{x_1, \ldots, x_q\}$ of $k+1$ different points.

This means, for example

$$\begin{aligned} M^q_1 &:= \{(x, \ldots, x) \in M^q\}, \\ M^q_{q-1} &= \{(x_1, \ldots, x_q) \in M^q \mid \exists i, j : i \neq j \text{ and } x_i = x_j\}, \end{aligned}$$

and clearly $M^q_1 \subset M^q_2 \subset \cdots \subset M^q_q = M^q$.

A straightforward calculation shows that the Schwartz kernel theorem gives the diagonal filtration an interpretation in terms of the support of distributions:

Proposition 4.7. An element $c \in B^q(\mathfrak{X}(M))$ is k-diagonal if and only if the support of its image under the Schwartz kernel map in $\mathrm{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM), \mathbb{R})$ is contained in M^q_k.

With this perspective, we can deduce:

Lemma 4.8. *For $U \subset M^q$, the assignments*

$$\begin{aligned} M^q \supset U &\mapsto \mathcal{B}^q(U) := \mathrm{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM)\big|_U, \mathbb{R}), \\ M^q_k \supset U &\mapsto \mathcal{B}^q_k(U) := \{c \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM), \mathbb{R}) : \mathrm{supp}\, c \subset U\}, \end{aligned}$$

constitute flabby cosheaves on M^q and M^q_k, respectively, where the extension maps are induced by the restriction maps of the section spaces.

Proof. Consider the sheaf of distributions $\mathcal{D}^q$ given by

$$\begin{aligned} M^q \supset U &\mapsto \mathcal{D}^q(U) := \mathrm{Hom}_{\mathbb{R}}\left(\Gamma_c(\boxtimes^q TM)\big|_U, \mathbb{R}\right), \\ M^q_k \supset U &\mapsto \mathcal{D}^q_k(U) := \{c \in \mathrm{Hom}_{\mathbb{R}}(\Gamma_c(\boxtimes^q TM), \mathbb{R}) : \mathrm{supp}\, c \subset U\} \end{aligned}$$

are soft. For the first sheaf, this follows since it is a module over a soft sheaf of rings, namely the sheaf of smooth functions on M^q. The second one is a restriction of the first sheaf to a closed subspace, hence soft (see [18, Chapter II, Thm 9.2, Thm 9.16]).

But $\mathcal{B}^q$ is exactly the precosheaf of compactly supported sections of the sheaf $\mathcal{D}^q$, and analogously for $\mathcal{B}^q_k$ and $\mathcal{D}^q_k$. By Proposition A.4, this implies that these precosheaves are flabby cosheaves. □

4.2. **Generalized good covers.** As we have seen, a component $C^k(\mathfrak{X}(M))$ cannot always be meaningfully understood to behave as (co-)sheaflike object over M, but rather should be understood as living over the cartesian product M^k. As such, we will need methods to compare different Cartesian powers $M, M^2, M^3, \ldots$ of M. One such tool we can use is the notion of a k-good cover in the sense of [12, Def 2.9]:

Definition 4.9. Let $k \geq 1$. An open cover $\mathcal{U}$ of M is *k-good* if:

i) Given k points $x_1, \ldots, x_k \in M$, there is a $U \in \mathcal{U}$ with $x_1, \ldots, x_k \in U$.
ii) All intersections of elements of $\mathcal{U}$ are diffeomorphic to a disjoint union of at most k copies of $\mathbb{R}^n$.

Remark 4.10. The k-good covers are, in a sense, finite approximations to so-called *Weiss covers*, which have property i) of the previous definition with no restriction on the number k, but without any replacement for property ii), so the sets in the cover may, a priori, be homologically wild. Weiss covers are heavily used in the theory of *factorization algebras,* which appear to have very strong ties to our setting, see [23, 13, 24].

For $k = 1$ this agrees with the usual notion of a good cover. Property i) of a k-good cover $\mathcal{U}$ is equivalent to, for all $q = 1, \ldots, k$, the set of cartesian powers $\{U^q : U \in \mathcal{U}\}$ being an open cover of M^q, making k-good covers useful tools in comparing data between the cartesian powers of M.

They also provide useful covers of the diagonals M^q_k, which we will show now. To this end, an auxiliary lemma:

Lemma 4.11. *Let $q \geq k \geq 1$ be integers, X a topological space and $U \subset X$ an open subset with finitely many connected components.*

If all connected components of U are contractible, then the connected components of U^q_k are contractible, and are precisely the sets $V \cap X^q_k$ for some connected component V of U^q with $V \cap X^q_k \neq \emptyset$.

In particular, if U is contractible, so is U^q_k.

Proof. Let $U_1, \ldots, U_s$ be the finitely many connected components of U, and V be a connected component of U^q. Then it is of the shape

$$V = U_{i_1} \times \cdots \times U_{i_q}$$

for some $i_1, \ldots, i_q \in \{1, \ldots, s\}$, not necessarily different.

By permutation and leaving out the connected components that do not contribute to V, we may assume that

$$V = U_1^{q_1} \times \cdots \times U_s^{q_s}$$

where $q_1, \ldots, q_s \geq 1$ and $\sum q_i = q$.

By assumption, $U_1, \ldots, U_s$ are all contractible, hence, for all $j = 1, \ldots, s$, there are points $y_j \in U_j$ and deformation retractions $F_j : U_j \times [0,1] \to U_j$ of U_j onto y_j.

The product $F_1^{q_1} \times \cdots \times F_s^{q_s}$ of these maps, restricted to the diagonal $[0,1] \subset [0,1]^q$, gives a deformation retraction

$$F : V \times [0,1] \to V$$

of V to the point

$$(\underbrace{y_1, \ldots, y_1}_{q_1 \text{ times}}, \ldots, \underbrace{y_i, \ldots, y_i}_{q_i \text{ times}} \ldots, \underbrace{y_s, \ldots, y_s}_{q_s \text{ times}}) \in V.$$

But if $V \cap X^q_k$ is nonempty, this map restricts to a deformation retraction of $V \cap X^q_k$, since, for all $j = 1, \dots, s$, all $l \in \mathbb{N}$, all $x_1, \dots, x_{q_s} \in U_j$ and $t \in [0,1]$:

$$|\{x_1, \dots, x_{q_s}\}| \leq l \implies |\{F_j(x_1,t), \dots, F_j(x_{q_s}, t)\}| \leq l,$$

and as a consequence, if $x \in V \cap X^q_k$, then $F(x,t) \in V \cap X^q_k$, in particular because all U_i are pairwise disjoint.

Hence, if $V \cap X^q_k$ is nonempty, this set is contractible and, in particular, connected. Hence, the finitely many connected components of U^q_k are all equal to some $V \cap X^q_k$ with V a connected component of X^q_k, and they are contractible.

If U itself was even contractible, then U^q has only one connected component, hence by the preceding argument U^q_k is contractible. □

Lemma 4.12. *Let $q \geq 2$ and $q \geq k \geq 1$ be integers and X a locally connected topological space. Define, for every $U \subset X$,*

$$U^q_k := U^q \cap X^q_k \subset X^q_k.$$

If $\mathcal{U}$ is a k-good open cover of X, then

$$\mathcal{U}^q_k := \{V \cap X^q_k \neq \emptyset : V \text{ a connected component of } U^q \text{ for some } U \in \mathcal{U}\}$$

is an open cover of X^q_k and all nonempty, finite intersections of sets in $\mathcal{U}^q_k$ are disjoint unions of contractible sets.

Proof. If $x \in X^q_k$, then the set of its components in the Cartesian product X^q contains at most k different points $x_1, \dots, x_k \in X$. But since $\mathcal{U}$ is k-good, there is some $U \in \mathcal{U}$ containing all $x_1, \dots, x_k$, and hence $x \in U^q$. Since x was arbitrary, this shows that $\mathcal{U}^q_k$ is an open cover of X^q_k.

Show now the statement about contractibility. Let $V \cap X^q_k, V' \cap X^q_k \in \mathcal{U}^q_k$ be two sets with nonempty intersection. By definition, V and V' are connected components of some U^q, U'^q with $U, U' \in \mathcal{U}$.

Since $\mathcal{U}$ is a k-good cover, Lemma 4.11 implies that all connected components of $U^q_k \cap (U')^q_k = (U \cap U')^q_k$ are contractible. But, also by Lemma 4.11, the sets $V \cap X^q_k$ and $V' \cap X^q_k$ are connected components of U^q_k and $(U')^q_k$, respectively.

Since X is locally connected, so is X^q_k, and as a consequence all connected components of open sets in X^q_k are closed and open. Then also $(V \cap X^q_k) \cap (V' \cap X^q_k)$ is closed and open in $U^q \cap (U')^q$, and hence must be a nonempty union of connected components of $U^q \cap (U')^q$. Since such connected components are contractible, all connected components of $V \cap V' \cap X^q_k = (V \cap X^q_k) \cap (V' \cap X^q_k)$ are, too.

By induction this extends to arbitrary finite intersections $(V_1 \cap \dots \cap V_r) \cap X^q_k$. The lemma is shown. □

The first part of the following theorem is Proposition 2.10 in [12]:

Theorem 4.13. *For every smooth manifold M, a k-good open cover exists. Further, if M is compact, then M admits finite k-good open covers.*

Proof. The existence of k-good open covers is shown in [12].

If M is compact, choose any k-good cover $\mathcal{U}$, then $\mathcal{U}^k$ is a cover of M^k, and since M^k is compact, there is a finite subcover $\tilde{\mathcal{U}} \subset \mathcal{U}$ so that $\tilde{\mathcal{U}}^k$ is a cover of M^k. Hence the set $\tilde{\mathcal{U}}$ fulfils property i) of being a k-good cover, and as a subset of a k-good cover, it also fulfils property ii). □

4.3. **The Čech-Bott-Segal double complex.**

Definition 4.14. For a smooth manifold M, an open cover $\mathcal{U}$ of M, and $k \geq 1$. We define the *k-th Čech-Bott-Segal (CBS) double complex* for the cover $\mathcal{U}$ as the following diagram:

$$
\begin{array}{ccccc}
\cdots & & \cdots & & \\
\uparrow & & \uparrow & & \\
\bigoplus_i \Delta_k C^3(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} \Delta_k C^3(\mathfrak{X}(U_i \cap U_j)) & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \\
\bigoplus_i \Delta_k C^2(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} \Delta_k C^2(\Gamma(\mathfrak{X}(U_i \cap U_j)) & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \\
\bigoplus_i \Delta_k C^1(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} \Delta_k C^1(\Gamma(\mathfrak{X}(U_i \cap U_j)) & \longleftarrow & \cdots
\end{array}
$$

The horizontal maps are given by the Čech differential associated to the cosheaf structure, whereas the vertical maps are given by the direct sum of Lie algebra differentials for the $C^\bullet(\mathfrak{X}(U_{i_1} \cap \cdots \cap U_{i_k}))$.

The *k-th skew-symmetrized CBS double complex* is the CBS double complex where the horizontal Čech complexes are replaced by their skew-symmetrized versions, see Remark A.7.

Our goal will be to understand the two spectral sequences that arise from taking the horizontal and vertical filtration of this complex. We should already note here that this is not the standard shape for a homology spectral sequence, since we are mixing a cohomological differential and a homological differential.

A priori, this means there is an ambiguity in defining the associated total complex, given by the choice of taking either direct sums or direct products on the relevant diagonals, since there may now be infinitely many nonzero terms on each such diagonal. The usual convergence theorems for the spectral sequences arising from horizontal and vertical filtration will, in general, not apply.

This is a significant problem, however, for *finite* covers $\mathcal{U}$, the skew-symmetrized complex fixes these issues:

Lemma 4.15. *Let $k \in \mathbb{N}$ and $\mathcal{U}$ a finite open cover of M. The k-th skew-symmetrized CBS double complex associated to $\mathcal{U}$ has only finitely many nonzero columns. In particular, it is bounded as a double complex.*

Proof. By finiteness of $\mathcal{U}$, there is a largest n such that there is a nonempty intersection $U_1 \cap \cdots \cap U_n$ with $U_i \neq U_j$ for $i \neq j$. Hence all columns in degree $> n$ vanish in the skew-symmetrized double complex. This concludes the proof. □

We begin with horizontal cohomology.

Proposition 4.16. Let $\mathcal{U}$ be a k-good cover of M and $q \geq 1$. If $k > q$, then we set the notation

$$\mathcal{U}_k^q := \mathcal{U}_q^q, \quad M_k^q := M^q, \quad \mathcal{B}_k^q := \mathcal{B}^q.$$

Then the Čech complex

$$\bigoplus_i \Delta_k B^q(\mathfrak{X}(U_i)) \leftarrow \bigoplus_{i,j} \Delta_k B^q(\mathfrak{X}(U_i \cap U_j)) \leftarrow \ldots$$

is isomorphic to the Čech complex associated to the flabby cosheaf

$$M_k^q \supset U \mapsto \mathcal{B}_k^q(U) = \{c \in \operatorname{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM), \mathbb{R}) : \operatorname{supp} c \subset M_k^q \cap U\}$$

with respect to the cover $\mathcal{U}_k^q$ of M_k^q, which was defined in Lemma 4.8.

The same statement holds for the skew-symmetrized Čech complex, and the isomorphism is equivariant under the natural permutation action of the symmetric group Σ_q.

Proof. Note first that $\mathcal{U}_k^q$ is a cover of M_k^q by Lemma 4.12.

The (restriction of the) Schwartz kernel maps (4.1) give us a family of isomorphisms $\{\phi_U : U \subset M \text{ open}\}$ as in Proposition 4.7, making for all open $U \subset V$ the following diagram commute:

$$\begin{array}{ccc} \Delta_k B^q(\mathfrak{X}(U)) & \longrightarrow & \Delta_k B^q(\mathfrak{X}(V)) \\ \big\downarrow{\scriptstyle \phi_U} & & \big\downarrow{\scriptstyle \phi_V} \\ \mathcal{B}_k^q(U_k^q) & \longrightarrow & \mathcal{B}_k^q(V_k^q) \end{array}$$

Hence, we have isomorphisms on the precosheaf data; this lifts to an isomorphism of the two Čech complexes.

This argument is independent from the choice of the standard or the skew-symmetrized Čech complex. Since the sets U_k^q are invariant under the natural Σ_q-action on M^q, both of the terms $\Delta_k B^q(\mathfrak{X}(U))$ and $\mathcal{B}_k^q(U_k^q)$ admit a Σ_k-action by permutation of vector fields. The Schwartz kernel map is equivariant with respect to this permutation, as one finds from the explicit formula of its dual map 4.2. □

Theorem 4.17. *Consider the k-th (skew-symmetrized) CBS double complex for a k-good cover $\mathcal{U}$.*

The cohomology of the q-th row is equal to $\Delta_k C^q(\mathfrak{X}(M))$ in degree zero, and trivial in all other degrees.

Proof. Fix the q-th row. By Proposition 4.16, the Čech complex in this row, associated to the cover $\mathcal{U}$ and the presheaf $U \mapsto \Delta_k B^q(\mathfrak{X}(U))$ over M, has the same homology as the Čech complex of the flabby cosheaf $U \mapsto \mathcal{B}_k^q(U)$ over M_k^q with respect to the cover $\mathcal{U}_k^q$. Flabby cosheaves have trivial Čech homology independent of the chosen cover by Proposition A.8, hence the homology is equal to $\Delta_k B^q(\mathfrak{X}(M))$ in zeroeth degree and zero else.

The isomorphism identifying the two complexes is equivariant with respect to the Σ_q-action on both spaces. The functor taking the complexes to its Σ_q-invariants is exact, as it arises from the action of a finite group in characteristic zero.

Hence, the skew-symmetrized complex also has trivial cohomology in nonzero degree, and in degree zero $(\Delta_k B^q(\mathfrak{X}(M)))^{\Sigma_q} = \Delta_k C^q(\mathfrak{X}(M))$. Since the skew-symmetrized complex is exactly the q-th row of the skew-symmetrized CBS complex, this concludes the proof. □

Corollary 4.18. Let $k \geq 1$ and assume there exists a finite, k-good cover $\mathcal{U}$ of M. Consider the spectral sequence $\{E_r^{p,q}, d_r\}$ associated to the skew-symmetrized k-th CBS double complex for $\mathcal{U}$, by filtering either along rows or columns.

This spectral sequence converges to $\Delta_k \tilde{H}^\bullet(\mathfrak{X}(M))$.

Proof. By Theorem 4.17, filtering by rows makes the spectral sequence collapse on the second page, with the indicated limit term $\Delta_k \tilde{H}^\bullet(\mathfrak{X}(M))$. Due to finiteness of $\mathcal{U}$ and Lemma 4.15, the skew-symmetrized double complex has bounded rows, and for such double complexes both filtrations yield spectral sequences which converge to the same cohomology, see [25, Chapter XV].

This shows the statement. $\square$

4.4. **Spectral sequences for diagonal cohomology.** Let us now investigate the spectral sequence arising from the k-th CBS double complex for a k-good cover, by considering the vertical differential first, i.e. the spectral sequence arising from the horizontal filtration. The cohomology among these vertical complexes amounts to calculating k-diagonal Lie algebra cohomology of $\mathfrak{X}(U)$, where the sets U are finite disjoint unions of $\mathbb{R}^n$.

To this end, let us first show that, for U such a disjoint union and k sufficiently big, there is no difference between k-diagonal cohomology of $\mathfrak{X}(U)$ and standard Lie algebra cohomology of $\mathfrak{X}(U)$.

Proposition 4.19. Let $1 \leq r \leq k$ and $U = \bigsqcup_{i=1}^r \mathbb{R}^n$. Then, the inclusion $\Delta_k C^\bullet(\mathfrak{X}(U)) \subset C^\bullet(\mathfrak{X}(U))$ induces an isomorphism

$$\Delta_k H^\bullet(\mathfrak{X}(U)) \cong H^\bullet(\mathfrak{X}(U)).$$

Proof. The construction in the proof of Proposition 3.17 restricts without change to the diagonally filtered complex. Specifically, the filtration $F^q\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))$ restricts to a filtration $F^q(\Delta_k\tilde{C}^\bullet)(\mathfrak{X}(\mathbb{R}^n))$, it is straightforward to check that the exact sequence

$$0 \to F^{q+1}\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^q\tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to \left(\bigoplus_{k_1+\cdots+k_r=q} C^\bullet_{(k_1)}(W_n)\otimes\cdots\otimes C^\bullet_{(k_r)}(W_n)\right)_{\mathrm{red}} \to 0$$

restricts to an exact sequence

$$0 \to F^{q+1}(\Delta_k\tilde{C}^\bullet)(\mathfrak{X}(\mathbb{R}^n)) \to F^q(\Delta_k\tilde{C}^\bullet)(\mathfrak{X}(\mathbb{R}^n)) \to \left(\bigoplus_{k_1+\cdots+k_r=q} C^\bullet_{(k_1)}(W_n)\otimes\cdots\otimes C^\bullet_{(k_r)}(W_n)\right)_{\mathrm{red}} \to 0,$$

and the image of the splitting $\beta_0^{(r)} : \left(C^\bullet_{(0)}(W_n)^{\otimes^r}\right)_{\mathrm{red}} \to \tilde{C}^\bullet(\mathfrak{X}(M))$ from the proof of Proposition 3.17 is contained in $\Delta_k\tilde{C}^\bullet(\mathfrak{X}(M))$. Hence,

$$\Delta_k H^\bullet(\mathfrak{X}(U)) \cong H^\bullet(W_n)^{\otimes^k} \cong H^\bullet(\mathfrak{X}(U)),$$

and all nontrivial cohomology classes in $H^\bullet(\mathfrak{X}(U))$ have representatives contained in $\Delta_k H^\bullet(\mathfrak{X}(U))$, hence the inclusion of complexes induces an isomorphism. This concludes the proof. $\square$

Consider now the CBS double complex for some k-good cover $\mathcal{U}$, and the cohomology with respect to the vertical differential. Every intersection in the cover $\mathcal{U}$ is diffeomorphic to a disjoint union of at most k copies of $\mathbb{R}^n$. Hence, Proposition 4.19 applies in every column and we can replace diagonal cohomology with standard Lie algebra cohomology. Hence, the first page of the spectral sequence assumes the following shape:

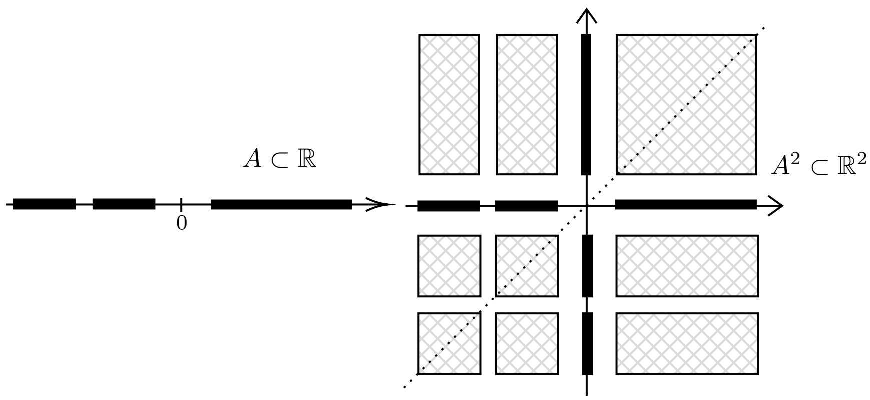

FIGURE 5. An illustration of Lemma 4.20: the set A has three connected components in $\mathbb{R}$, so its square $A^2 \subset \mathbb{R}^2$ has $9 = 3^2$, all arising by taking products of connected components of A. The products of a connected component of A with itself are exactly the connected components of A^2 which intersect the diagonal in $\mathbb{R}^2$.

$$\begin{array}{ccccc}
\cdots & & \cdots & & \\
\bigoplus_i H^3(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} H^3(\mathfrak{X}(U_i \cap U_j)) & \longleftarrow & \cdots \\
\bigoplus_i H^2(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} H^2(\Gamma(\mathfrak{X}(U_i \cap U_j)) & \longleftarrow & \cdots \\
\bigoplus_i H^1(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} H^1(\Gamma(\mathfrak{X}(U_i \cap U_j)) & \longleftarrow & \cdots
\end{array}$$

We state the following simple lemma without proof (see also Figure 4.4):

Lemma 4.20. *Given an subset $U \subset M$ and a number $q \geq 2$, the connected components of the set U^q intersect M^q_{q-1} if and only if U has $\geq q$ connected components. More precisely, the connected components of U^q that do not intersect M^q_{q-1} are exactly the Cartesian products of q different connected components of U.*

Proposition 4.21. Let $q \geq k \geq 1$. Consider the k-th CBS complex with respect to a k-good cover $\mathcal{U}$, and its spectral sequence $\{E_r^{p,q}, d_r\}$ arising from the horizontal filtration. Recall the sets $\mathcal{U}^q_k$ defined in Lemma 4.12.

Then the complex in the q-th row $E_1^{\bullet,q}$ is naturally isomorphic to a direct sum of relative Čech complexes, namely

$$\begin{aligned} & \check{C}_\bullet(\mathcal{U};\mathcal{H}^q) \oplus \check{C}_\bullet\left(\mathcal{U}_2^2,\mathcal{U}_1^2;\bigoplus_{\substack{q_1+q_2=q\\ q_1,q_2>0}}\mathcal{H}^{q_1}\otimes\mathcal{H}^{q_2}\right)^{\Sigma_2} \\ & \oplus \quad \dots \\ & \oplus \check{C}_\bullet\left(\mathcal{U}_k^k,\mathcal{U}_{k-1}^k;\bigoplus_{\substack{q_1+\dots+q_k=q\\ q_1,\dots,q_k>0}}\mathcal{H}^{q_1}\otimes\cdots\otimes\mathcal{H}^{q_k}\right)^{\Sigma_k}. \end{aligned} \tag{4.3}$$

Here, the symmetric groups $\Sigma_2,\dots,\Sigma_k$ act by simultaneous skew-symmetric permutation of the factors $U_1\times\cdots\times U_n$ of any Cartesian product and the tensor factors of $\mathcal{H}^{q_1}\otimes\cdots\otimes\mathcal{H}^{q_k}$.

The same statement holds for the skew-symmetrized CBS complex, when the Čech complexes in 4.3 are replaced by their skew-symmetrized versions.

Proof. We will first show how we embed a direct summand from $E_1^{\bullet,q}$ into the specified Čech complexes.

Let $U = U_1\cap\cdots\cap U_l$ with $U_1,\dots,U_l\in\mathcal{U}$, and assume U has $r\le k$ connected components $C_1,\dots,C_r\subset U$. Then a choice of ordering on the C_i gives rise to an isomorphism

$$H^\bullet(\mathfrak{X}(U)) = H^\bullet(\mathfrak{X}(C_1))\otimes\cdots\otimes H^\bullet(\mathfrak{X}(C_r)). \tag{4.4}$$

Fix a direct summand

$$H^{q_1}(\mathfrak{X}(C_1))\otimes\cdots\otimes H^{q_r}(\mathfrak{X}(C_r))\subset\tilde{H}^\bullet(\mathfrak{X}(U)), \tag{4.5}$$

with $q := \sum_{i=1}^r q_i$.

Assume that exactly m of the numbers $q_1,\dots,q_r$ are nonzero, w.l.o.g. $q_1,\dots,q_m$. Then we have the natural isomorphism

$$H^{q_1}(\mathfrak{X}(C_1))\otimes\cdots\otimes H^{q_r}(\mathfrak{X}(C_r))\cong H^{q_1}(\mathfrak{X}(C_1))\otimes\cdots\otimes H^{q_m}(\mathfrak{X}(C_m)). \tag{4.6}$$

Consider now the Čech complex

$$\check{C}_\bullet\left(\mathcal{U}_k^k;\bigoplus_{\substack{q_1+\dots+q_k=q\\ q_1,\dots,q_k>0}}\mathcal{H}^{q_1}\otimes\cdots\otimes\mathcal{H}^{q_k}\right).$$

In this complex, there is for every permutation $\sigma\in\Sigma_m$ a direct summand associated to the connected component $C_{\sigma(1)}\times\cdots\times C_{\sigma(m)}$ of the set U^m which equals

$$H^{q_{\sigma(1)}}(W_n)\otimes\cdots\otimes H^{q_{\sigma(m)}}(W_n)\subset C_\bullet(\mathcal{U}_m^m,\mathcal{H}^{q_{\sigma(1)}}\otimes\cdots\otimes\mathcal{H}^{q_{\sigma(m)}}). \tag{4.7}$$

Since $C_i\neq C_j$ for $i\neq j$, the sets $C_{\sigma(1)}\times\cdots\times C_{\sigma(m)}$ do not intersection M_{k-1}^k, so by Lemma 4.20 these direct summands do not show up in the subcomplex $C_\bullet(\mathcal{U}_{m-1}^m,\mathcal{H}^{\otimes m})$. Hence, these direct summands do not vanish in the relative complex $C_\bullet(\mathcal{U}_m^m,\mathcal{U}_{m-1}^m,\mathcal{H}^{\otimes^m})$.

Since $H^\bullet(W_n)\cong H^\bullet(\mathfrak{X}(C_j))$ for all j, the term (4.7) equals (4.5) up to permutation. Note further that a different choice of ordering in (4.4) corresponds to a skew-symmetric permutation of the tensor factors. Hence we may naturally identify (4.5) and the invariants associated to (4.7).

This construction extends to all of $\tilde{H}^\bullet(\mathfrak{X}(U))$ and hence to all of $E_1^{\bullet,q}$. Since every direct summand of the relative Čech complexes in (4.3) can be completely decomposed into terms of the form (4.7), this construction is surjective onto (4.3) and, as graded vector spaces, we can identify $E_1^{\bullet,q}$ with the proposed direct sum of Čech complexes (4.3).

Now it remains to show that this is actually an identification of chain complexes, i.e. that the differentials of the spectral sequence are mapped to the relative Čech differentials. Note that the differential of $E_1^{\bullet,\bullet}$ is itself induced by the Čech differential of the precosheaf

$$\tilde{H}^\bullet(\mathfrak{X}(U)) \to \tilde{H}^\bullet(\mathfrak{X}(V)),$$

so to show that the differentials of both Čech complexes agree, it suffices to show that the extension maps of the precosheaves $H^k(\mathfrak{X}(U)) \to H^k(\mathfrak{X}(V))$ are mapped to the extension maps of the associated cosheaves $\mathcal{H}, \mathcal{H}^{\otimes 2}, \dots, \mathcal{H}^{\otimes k}$.

Consider any two open sets $U \subset V$, both equal to disjoint unions of open balls

$$U = C_1 \cup \cdots \cup C_k, \quad V = D_1 \cup \cdots \cup D_{k'}$$

We want to examine the extension map

$$\begin{aligned} H^\bullet(\mathfrak{X}(U)) &\cong H^\bullet(\mathfrak{X}(C_1)) \otimes \cdots \otimes H^\bullet(\mathfrak{X}(C_k)) \\ &\to H^\bullet(\mathfrak{X}(D_1)) \otimes \cdots \otimes H^\bullet(\mathfrak{X}(D_{k'})) \cong H^\bullet(\mathfrak{X}(V)). \end{aligned} \tag{4.8}$$

If there are two connected components of U which lie in a single connected component of V, w.l.o.g. $C_1, C_2 \subset D_1$, by Corollary 3.18 the extension map

$$H^{q_1}(\mathfrak{X}(C_1)) \otimes H^{q_2}(\mathfrak{X}(C_2)) \to H^{q_1+q_2}(\mathfrak{X}(C_1))$$

assigns two cohomology classes on C_1 and C_2 to their wedge product, and by Corollary 2.25, this wedge product is zero if q_1, q_2 are both nonzero.

Hence, the only nonzero extension maps on direct summands within (4.8) are, up to additional tensor factors of degree zero, of the form

$$\begin{aligned} H^{q_1}(\mathfrak{X}(C_{r_1})) \otimes \cdots \otimes H^{q_k}(\mathfrak{X}(C_{r_l})) &\to H^{q_1}(\mathfrak{X}(D_{r_1})) \otimes \cdots \otimes H^{q_k}(\mathfrak{X}(D_{r_l})), \\ [c_1] \otimes \cdots \otimes [c_l] &\mapsto [\iota_{C_{r_1}}^{D_{r_1}} c_1] \otimes \cdots \otimes [\iota_{C_{r_k}}^{D_{r_k}} c_l]. \end{aligned} \tag{4.9}$$

with $C_{r_i} \subset D_{r_i}$, $r_i \neq r_j$ for $i \neq j$ and $q_1, \dots, q_l > 0$.

But this is exactly the extension map of the Cartesian product

$$C_{r_1} \times \cdots \times C_{r_l} \subset D_{r_1} \times \cdots \times D_{r_l} \tag{4.10}$$

within the relative Čech complex of $(\mathcal{U}^l, \mathcal{U}^l_{l-1})$, and every extension map in this relative Čech complex arises through an inclusion of connected components of the form (4.10).

Additional tensor factors of degree zero in the domain or codomain of (4.9) yield equivalent maps, via the isomorphism

$$V \otimes_{\mathbb{R}} \mathbb{R} \cong V$$

for arbitrary $\mathbb{R}$-vector spaces $\mathbb{R}$.

This shows that the extension maps and thus the Čech differentials are respected by our construction. The proof is done.

□

If M is not orientable, there does not seem to be anything further we can immediately extract from these relative Čech complexes.

However, if M is orientable, then Proposition 3.22 implies that all the cosheaves $\mathcal{H},\dots,\mathcal{H}^k$ are constant, and hence, because $\mathcal{U}^k_{k-1}=\{U\cap M^k_{k-1} : U\in\mathcal{U}^k\}$:

$$\check{H}_\bullet(\mathcal{U};\mathcal{H}^q)\cong \check{H}_\bullet(M_\mathcal{U})\otimes H^q(W_n),$$

where the Čech homology on the right hand side is the traditional Čech homology for M with respect to the cover $\mathcal{U}$, and

$$\check{H}_\bullet(\mathcal{U}^k_k,\mathcal{U}^k_{k-1};\mathcal{H}^{q_1}\otimes\cdots\otimes\mathcal{H}^{q_k}) \cong \check{H}_\bullet\left((M^k)_{\mathcal{U}^k},(M^k_{k-1})_{M^k_{\mathcal{U}^k}}\right)\otimes H^{q_1}(W_n)\otimes\cdots\otimes H^{q_k}(W_n),$$

where the relative Čech homology on the right hand side is the traditional relative Čech homology with real coefficients for the pair (M^k,M^k_{k-1}) with respect to the cover $\mathcal{U}^k$, see [26, Chapter IX].

Lemma 4.22. *If $\mathcal{U}$ is a finite k-good cover for M, then the (relative) Čech homologies $\check{H}_\bullet(M_\mathcal{U})$ (and $\check{H}_\bullet(\mathcal{U}^l_l,\mathcal{U}^l\cap M^l_{l-1})$ for $2\le l\le k$) are isomorphic to (relative) singular homology.*

Proof. If $\mathcal{U}$ is finite, then all the complexes $\check{C}_\bullet(\mathcal{U})$ and $\check{C}_\bullet(\mathcal{U}^k_k,\mathcal{U}^k\cap M^k_{k-1})$ are finite complexes of finite-dimensional vector spaces.

Dualizing is an exact functor in this setting, and the dual of these complexes are immediately the corresponding (relative) complexes in Čech *cohomology*.

Now, because $\mathcal{U}$ is an k-good cover and by Lemma 4.12, all the sets $\mathcal{U},\mathcal{U}^2_2,\dots,\mathcal{U}^k_k$ are *Leray covers* for the constant sheaves on $M,M^2,\dots,M^k$, in the sense that their sheaf cohomology is isomorphic to the Čech cohomology with respect to this cover, and the set $\mathcal{U}^2_1,\dots,\mathcal{U}^k_{k-1}$ are Leray covers for the constant sheaves on $M^2_1,\dots,M^k_{k-1}$ (see also [27, Chapter VI.D, Theorem 4]).

Hence these cohomologies agree with (relative) Čech cohomology of the space M (or the pair (M^q,M^q_{q-1})), which is itself well-known to be (relative) singular cohomology.

Since the dual of (relative) singular cohomology in this setting is (relative) singular homology, the statement is shown. □

Corollary 4.23. Let M be an orientable manifold which admits a finite, k-good open cover (e.g. M compact).

Then there exists a cohomological spectral sequence $\{E^{\bullet,\bullet}_r,d_r\}$ which converges to $\Delta_k H^\bullet(\mathfrak{X}(M))$ whose term $E^{p,q}_2$ in its second page equals

$$\begin{aligned} &(H_{-p}(M)\otimes H^q(W_n))\\ \oplus &\bigoplus_{\substack{q_1+q_2=q\\ q_i>0}}\left(H_{-p}(M^2,M^2_1)\otimes H^{q_1}(W_n)\otimes H^{q_2}(W_n)\right)^{\Sigma_2}\\ \oplus &\ \cdots\\ \oplus &\bigoplus_{\substack{q_1+\cdots+q_k=q\\ q_i>0}}\left(H_{-p}(M^k,M^k_{k-1})\otimes H^{q_1}(W_n)\otimes\cdots\otimes H^{q_k}(W_n)\right)^{\Sigma_k}. \end{aligned}\tag{4.11}$$

Here, a permutation $\sigma\in\Sigma_r$ acts by simultaneous, skew-symmetric permutation of the Cartesian factors of M^k and the tensor factors $H^q(W_n)$.

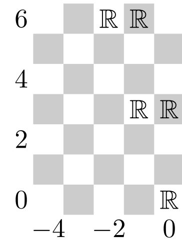

FIGURE 6. The spectral sequence for 2-diagonal Lie algebra cohomology for $\mathfrak{X}(S^1)$. For the k-diagonal spectral sequences for $k \geq 2$, this pattern continues into the upper-left direction.

Proof. Under the reflection $p \mapsto -p$, the spectral sequence of the skew-symmetrized Čech complex from Proposition 4.21 becomes a cohomological spectral sequence and, by Lemma 4.22 has this second page, using any k-good cover of M.

Because of Theorem 4.13, we can choose the cover to be finite. Hence, Corollary 4.18 implies that the spectral sequence must converge to $\Delta_k H^\bullet(\mathfrak{X}(M))$. This concludes the statement. □

Remark 4.24. These spectral sequences differ from the ones stated in [9], but only insofar as they consider the quotient complexes $\Delta_k C^\bullet(\mathfrak{X}(\mathbb{R}^n)/\Delta_{k-1}C^\bullet(\mathfrak{X}(\mathbb{R}^n)$ rather than the diagonal complexes themselves. This essentially gives one spectral sequence for every row in (4.11).

For $k \geq q+1$, we have $\Delta_k H^q(\mathfrak{X}(M)) = H^q(\mathfrak{X}(M))$ so in principle, these spectral sequences can be used to calculate the full Lie algebra cohomology of $\mathfrak{X}(M)$, degree by degree.

In particular, since we know that the nontrivial cohomology of W_n is contained within the degrees $q = 2n+1, \ldots, 2n+n^2$ and the relative cohomology of the pair (M^k, M^k_{k-1}) in degrees $\leq nk$, we have the following:

Corollary 4.25. For all smooth manifolds M that admits k-good open covers for all $k \in \mathbb{N}$, and all $n \geq 0$, the Gelfand-Fuks cohomology $H^n(\mathfrak{X}(M))$ is finite-dimensional. If $\dim M \geq k \geq 1$, then $H^k(\mathfrak{X}(M)) = 0$.

Example 4.26. If $M = S^1$, one finds in relative singular homology:

$$H_r\left((S^1)^k, (S^1)^k_{k-1}\right) = \begin{cases} \mathbb{R}^{(k-1)!} & \text{if } r = k, k-1, \\ 0 & \text{else,} \end{cases}$$

where the copies of $\mathbb{R}$ in $\mathbb{R}^{(k-1)!}$ are enumerated by permutations of the $(k-1)$-th symmetric group, and the invariant space under the action of the k-th symmetric group Σ_k is one-dimensional.

Using this, we find that in the spectral sequence for k-diagonal cohomology, there is only ever at most a single nontrivial term on every diagonal $p+q = \text{const}$, and those only exist on the diagonals $p+q = 0, 2, 3, 5, 6, 8, 9, \ldots$. From lacunary arguments, one concludes that all differentials beyond the second page must be trivial. Hence, for all $k \geq 0$, we have

$$H^k(\mathfrak{X}(S^1)) = \begin{cases} \mathbb{R} & \text{if } k \neq 1 \text{ mod } 3, \\ 0 & \text{else.} \end{cases}$$

4.5. **A topological model for Gelfand-Fuks cohomology.** Finally, we want to mention the historical conclusion to the investigation of Gelfand-Fuks cohomology, namely the existence of a topological model for Gelfand-Fuks cohomology for certain smooth manifolds M, i.e. a topological space X such that $H^\bullet_{\text{sing}}(X) \cong H^\bullet_{\text{CE}}(\mathfrak{X}(M))$. This has been carried out in [4], originally, and is also presented in [9]. We have not been able to substantially add to the presentation or proofs in the given literature, so we will simply state the most important theorems and hint at the ideas behind their proofs.

We begin with a topological model for the cohomology of formal vector fields $H^\bullet(W_n)$, as studied in Section 2. Consider the complex Grassmanian $G(n, 2n)$ of n-dimensional subspaces of $\mathbb{C}^{2n}$ as a smooth manifold. Its cohomology ring up to dimension $\leq 2n$ is freely generated by generators $\Psi_2, \Psi_4, \ldots, \Psi_{2n}$, one in every even dimension.

As a CW-complex, it has a natural cell decomposition into its *Schubert cells*, which are complex manifolds, and hence have even real dimension. As a consequence, the cohomology of the *$2n$-skeleton* $\tilde{B} \subset G(n, 2n)$, the space of all CW-cells up to dimension $\leq 2n$, has the same generators, but with the relation that products of degree $> 2n$ vanish.

We recognize this algebra as appearing in the rows of the spectral sequence from Theorem 2.24, which calculates the cohomology of the formal vector fields W_n. Indeed, in analogy to the theory of equivariant differential forms and principal bundles, one may consider this as the *basic* part of some object, over which the Lie group $GL_n(\mathbb{R})$ acts on the "fibres" – in correspondence to the cohomology of $\mathfrak{gl}_n(\mathbb{R})$ appearing in the columns of the spectral sequence. Bott and Segal make these notions precise by defining what they call *G-cochain algebras*, an algebraic generalization of G-principal fibre bundles.

Because CW-complexes and subcomplexes thereof are good pairs in the sense of algebraic topology, $\tilde{B}$ admits an open neighbourhood in $G(n, 2n)$ with equal singular cohomology. We denote this smooth manifold by B. Recall also the *tautological bundle* $V(n, 2n) \to G(n, 2n)$, where the total space $V(n, 2n)$ is given by collections of n linearly independent vectors of $2n$, and its projection to $G(n, 2n)$ equals the projection of such a collection to the subspace it spans in $\mathbb{C}^{2n}$.

Finally, we are prepared to state the topological model:

Theorem 4.27. *Let B_n be an open neighbourhood of the $2n$-skeleton of the Grassmannian $G(n, 2n)$ that deformation retracts onto this skeleton. Let further $F_n \to B_n$ be the restriction of the tautological, principal $GL_n(\mathbb{C})$-bundle $V(n, 2n) \to G(n, 2n)$ to B_n.*

Then there is a $O(n)$-equivariant zig-zag of quasi-isomorphisms between $C^\bullet(W_n)$ and $C^\bullet_{\text{dR}}(F_n)$. In particular:

$$H^\bullet(W_n) \cong H^\bullet_{\text{sing}}(F_n).$$

Proof sketch. Both $C^\bullet(W_n)$ and $\Omega^\bullet_{\text{dR}}(F_n)$ are G-cochain algebras with $G = GL_n(\mathbb{R})$ and $G = U(n)$, respectively, and they both admit *standard connections*, as defined in [4]. These connections allow the cohomologies of the G-cochain algebras to be reduced to calculation of the cohomology of the respective *basic subcomplexes*. For $C^\bullet(W_n)$, this is the relative complex $C^\bullet(W_n, \mathfrak{gl}_n(\mathbb{R}))$, for $\Omega^\bullet(F_n)$ this is the de Rham complex of the base manifold $\Omega^\bullet_{\mathrm{dR}}(B_n)$. The cohomology of these subcomplexes are identical for both G-cochain algebras, and equal to the singular cohomology of the space B_n.

Consider the *Sullivan minimal cochain algebra* $M^\bullet$ of the complex $\Omega^\bullet_{\mathrm{dR}}(B)$ (originally due to [28], see also [29, Prop. 12.2]). This embeds into the basic subcomplex of both cochain algebras by quasi-isomorphisms. By reducing the fiber groups of both G-cochain algebras to their common intersection $O(n) = U(n) \cap GL_n(\mathbb{R})$, this zig-zag of quasi-isomorphisms can be lifted to a zig-zag of $O(n)$-equivariant quasi-isomorphisms $C^\bullet(W_n)$ and $\Omega^\bullet(F_n)$. This concludes the sketch. □

Now, just as we treated W_n in our previous calculations as a local model which globally glues to Gelfand-Fuks cohomology, it turns out that the topological model for $H^\bullet(W_n)$ globally glues to a topological model for $H^\bullet(\mathfrak{X}(M))$. This model appears as an infinite-dimensional mapping space, and as such the local-to-global analysis is, while spiritually similar to our proof of Corollary 4.23, quite a lot more involved.

Bott and Segal construct, analogous to Corollary 4.23, a spectral sequence that calculates k-diagonal Gelfand-Fuks cohomology for every $k \geq 1$. In contrast to us, they work with a single good cover (rather than k-good) of M, at the cost of having to construct non-standard Čech complexes. We remark that, as in Corollary 4.23, the existence of a *finite* good cover is necessary for Bott and Segal to resolve convergence issues of the arising spectral sequence.

By carefully analyzing how the cohomology of mapping spaces localizes, the zig-zag of quasi-isomorphisms in Theorem 4.27 applied to the terms within the spectral sequences lifts to the desired a zig-zag of quasi-isomorphisms between simplicial cohomology of the topological model and Gelfand-Fuks cohomology.

Finally, one arrives at the following:

Theorem 4.28. *Let M be a Riemannian manifold that admits a finite cover by geodesically convex sets (e.g. M compact or the interior of a compact manifold with boundary).*

If $P \to M$ is the natural principal O_n-bundle over M, let $E \to M$ be the associated $O(n)$-bundle with fibre F_n as constructed in Theorem 4.27. Consider $\Gamma(E)$ as a topological space with its standard Fréchet topology.

Then there is a zig-zag of cohomology equivalences between the Gelfand-Fuks cochains $C^\bullet(\mathfrak{X}(M))$ and singular cochains $C^\bullet_{\mathrm{sing}}(\Gamma(E))$. In particular:

$$H^\bullet(\mathfrak{X}(M)) \cong H^\bullet_{\mathrm{sing}}(\Gamma(E)).$$

Appendix A. Cosheaves and Čech homology

A.1. **Basic definitions.** In this appendix, we will recall some useful statements about cosheaf theory from [18].

In this section, fix a topological space M with topology $\mathcal{U}$.

Definition A.1. [18, Chapter V.1] Let M be a topological space.

i) A *precosheaf* (of abelian groups) $\mathcal{P}$ on M is a covariant functor from the category of open sets of M, morphisms given by inclusions, into the category of abelian groups. Given an inclusion $U \subset V$ of open sets, we denote the associated mapping $\mathcal{P}(U) \to \mathcal{P}(V)$ by ι_U^V, called the *extension map* from U to V of the precosheaf $\mathcal{P}$.

ii) A *cosheaf* is a precosheaf $\mathcal{P}$ with the property that for every open cover $\mathcal{U}$ of an open set $U \subset M$, the sequence

$$\bigoplus_{i,j} \mathcal{P}(U_i \cap U_j) \to \bigoplus_i \mathcal{P}(U_i) \to \mathcal{P}(U) \to 0$$

is exact, where the maps are given by

$$(a_{ij})_{i,j} \mapsto \left(\sum_j \iota_{U_i \cap U_j}^{U_i} (a_{ij} - a_{ji}) \right)_i , \quad (b_i)_i \mapsto \sum_i \iota_{U_i}^U b_i.$$

iii) A *morphism of (pre-)cosheaves* is a natural transformation between the functors defining the (pre-)cosheaves.

We will implicitly assume that all our precosheaves take values in Ab. In [18], the precosheaves are assumed to take values in the category of modules over abelian groups, but this implies our setting.

One important cosheaf is the *constant cosheaf*, which we cite from [18, Section V.1].

Example A.2. If M is locally connected and A is some abelian group, then one defines the *constant cosheaf* A over M as the precosheaf which assigns to an open $U \subset M$ the abelian group $V^{\pi_0(U)}$, where $\pi_0(U)$ is the set of connected components of U.

The extension maps of this constant cosheaf corresponding to an inclusion $U \subset V$ is then given by taking the sum among all elements which map from different connected components of U into the same connected component of V.

This is, in the appropriate sense, dual to the more well-known constant sheaf over M.

Definition A.3. Given a cosheaf, we call it *flabby* if all its extension maps are injective.

Proposition A.4 ([18], Chapter 5, Proposition 1.6)**.** If P is a soft sheaf, then the compactly supported sections of P have the structure of a flabby cosheaf.

A.2. **Čech homology of cosheaves.** We borrow some further notions from [18, Chapter VI, Section 4]:

Definition A.5. Let S be a precosheaf over M, and $\mathcal{U} = \{U_\alpha\}$ an open cover of M. Write $\alpha := (\alpha_1, \dots, \alpha_{p+1})$ for the p-simplex defined by a collection of indices α_i, and write

$$U_\alpha := U_{\alpha_1} \cap \cdots \cap U_{\alpha_{p+1}}.$$

Further, define for a p-simplex α and a number $i \in \{1, \dots, p+1\}$ the $(p-1)$-simplex $\alpha^{(i)}$ arising by removing the i-th index from α.

For all $p \geq 0$, we define the space of *Čech p-chains for S associated to the cover $\mathcal{U}$* as

$$\check{C}_p(\mathcal{U}; S) := \bigoplus_{\alpha=(\alpha_1,\dots,\alpha_{p+1})} S(U_\alpha).$$

We may then express elements $c \in \check{C}_p(\mathcal{U}; S)$ as finite formal linear combinations

$$c = \sum_\alpha c_\alpha \cdot \alpha, \quad c_\alpha \in S(U_{\alpha_1} \cap \cdots \cap U_{\alpha_{p+1}}), \tag{A.1}$$

so that only finitely many c_α are nonzero.

The *Čech differential* $\check{\partial} : \check{C}_p(\mathcal{U}; S) \to \check{C}_{p-1}(\mathcal{U}; S)$ via

$$\check{\partial}(c_\alpha \cdot \alpha) := \sum_{i=1}^{p+1} (-1)^{i-1} \left(\iota_{U_\alpha}^{U_{\alpha^{(i)}}} c_\alpha \right) \cdot \alpha^{(i)}.$$

This defines a complex structure on $\check{C}_\bullet(\mathcal{U}; P) := \bigoplus_{p \geq 0} \check{C}_p(\mathcal{U}; S)$, and we denote its homology in degree p by $\check{H}_p(\mathcal{U}; S)$, the *p-th Čech homology group associated to the cover $\mathcal{U}$ and the precosheaf S.*

With respect to refinement, the set of open covers on M becomes a directed set, and in this sense the set of Čech homologies with respect to open covers becomes an inverse system, so that one can define *Čech homology of M and S* as

$$\check{H}_\bullet(M; S) := \varprojlim \check{H}_\bullet(\mathcal{U}; S).$$

Remark A.6. If the cosheaf S is the constant cosheaf $U \mapsto \mathbb{R}$, then this definition equals the definition of the conventional Čech homology of a topological space.

Remark A.7. It is sometimes useful to instead consider the *skew-symmetrized* Čech complex, i.e. the subcomplex $\check{C}^a_\bullet(\mathcal{U}; S) \subset \check{C}_\bullet(\mathcal{U}; S)$ defined by the skew-symmetrized cochains in the following sense:

The symmetric group Σ_p acts on multiindices α of length p by permutation of the entries, and we denote this permutation by $\sigma \cdot \alpha$.

Recall now the notation from (A.1). If $c = \sum_\alpha c_\alpha \cdot \alpha$ and $\alpha = (\alpha_1, \ldots, \alpha_p)$ is one of the multiindices, we call c *skew-symmetric* if

$$c_{\sigma \cdot \alpha} = \operatorname{sign}(\sigma) \cdot c_\alpha \quad \forall \alpha.$$

Consider the embedding

$$\check{C}^a_\bullet(\mathcal{U}; S) \to \check{C}_\bullet(\mathcal{U}; S).$$

In the dual setting of Cech cohomology of a sheaf, the corresponding dual morphism is shown to be a quasi-isomorphism in [30, Section 3.8], and this proof is straightforwardly dualized to the cosheaf setting.

Hence, this embedding, too, is a quasi-isomorphism.

Proposition A.8 ([18], Chapter VI, Corollary 4.5)**.** If S is a flabby cosheaf over M, then for every cover $\mathcal{U}$ of M,

$$\check{H}_p(\mathcal{U}; S) = \begin{cases} S(M) & \text{if } p = 0, \\ 0 & \text{else.} \end{cases}$$

A.3. **Relative Čech homology.**

Definition A.9. Let $A \subset M$ be a topological subspace, and let S and T be precosheaves on A and M, respectively.

i) We define the precosheaf S^M on M by

$$S^M(U) := S(U \cap A) \quad \forall U \subset M,$$

with the obvious extension maps.

ii) A *cover* of the pair (M, A) is a pair $(\mathcal{U}, \mathcal{U}_0)$, where $\mathcal{U}$ is a covering of M and $\mathcal{U}_0 \subset \mathcal{U}$ is a covering of A.

iii) Assume there is a monomorphism of precosheaves

$$\eta : S^M \to T.$$

Then, this induces an injective map

$$\check{C}_\bullet(\mathcal{U}_0 \cap A, S) = \check{C}_\bullet(\mathcal{U}_0, S^M) \to \check{C}_\bullet(\mathcal{U}, T).$$

Then, the *relative complex associated to* S, T, *and* $(\mathcal{U}, \mathcal{U}_0)$ is defined as the cokernel of this map, and denote by

$$\check{C}_\bullet(\mathcal{U}, \mathcal{U}_0; S, T).$$

Correspondingly, the homology arising rom this is called *relative Čech homology associated to the precosheaves* S *and* T*, and the cover* $(\mathcal{U}, \mathcal{U}_0)$, and is denoted by $\check{H}_\bullet(\mathcal{U}, \mathcal{U}_0; S, T)$.

If we choose for S and T the constant cosheaves (see Example A.2) on X and A, and $\mathcal{U}_0 = \{U \cap A : U \in \mathcal{U}, U \cap A \neq \emptyset\}$, then this is simply the standard relative Čech complex $\check{C}_\bullet(\mathcal{U}, \mathcal{U} \cap A)$ for the pair (M, A) and cover $\mathcal{U}$.

A.4. **Cosheaves on a base.** While the concept of sheaves on a base is well-studied, there are no remarks about cosheaves on a base in the literature. Luckily, in this category, there is no additional work to do.

Definition A.10. Let $\mathcal{B}$ be a topological base of M. In the following, view $\mathcal{B}$ as a subcategory of the category of open sets of M.

i) A *precosheaf* S *on* $\mathcal{B}$ is a covariant functor from $\mathcal{B}$ to the category of abelian groups. We denote the image of $U \in \mathcal{B}$ as $S(U)$ and the arising extension maps for $U \subset V \in \mathcal{B}$ by ι_U^V.

ii) Choose for any $U \in \mathcal{B}$ an open cover $\{U_i\}_{i \in I}$ by elements in $\mathcal{B}$, and for every $i, j \in I$ an open cover $\{V_{ij,k}\}_{k \in K}$ of $U_i \cap U_j$ by elements in $\mathcal{B}$. We call a precosheaf S on $\mathcal{B}$ a *cosheaf on* $\mathcal{B}$ if, for all such choices, the following sequence is exact:

$$0 \leftarrow P(U) \leftarrow \bigoplus_i P(U_i) \leftarrow \bigoplus_{ijk} P(V_{ij,k}).$$

iii) A morphism of (pre-)cosheaves on $\mathcal{B}$ is a natural transformation of the functors defining the (pre-)cosheaves.

The sequence is the analogue of the cosheaf condition, just restricted to only working with information on $\mathcal{B}$. This is precisely the dual of the well-studied concept of sheaves on a base, by viewing Ab-valued cosheaves as $\mathrm{Ab}^{\mathrm{op}}$-valued sheaves.

Theorem A.11. *Given a topological space* M *and a topological base* $\mathcal{B}$ *of* M*. An Ab-valued cosheaf on* $\mathcal{B}$ *extends, up to cosheaf isomorphism, uniquely to a cosheaf on* M*.*

A morphism between two cosheaves on a base $\mathcal{B}$ *of* M *extends uniquely to a morphism between the induced cosheaves on* M*.*

Proof. The following proof is due to [31]. The analogue statement for $\mathcal{C}$-valued sheaves is true whenever $\mathcal{C}$ is a complete category (see [32] for a proof in the category of modules over a ring). However, since Ab is a cocomplete category, $\mathrm{Ab}^{\mathrm{op}}$ is a complete category. This proves the statement.[1] □

Appendix B. Representation theory and cohomology of $\mathfrak{gl}_n$

We need a couple results about the representation theory of $\mathfrak{gl}_n(\mathbb{R})$. All things we need are also presented in [9], but for the results where this is possible, we cite independent results and literature.

Definition B.1. Let $V = \mathbb{R}^n$ be a finite-dimensional vector space. The Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ acts on V by matrix multiplication, and on V^* by the appropriate pullback multiplied with a minus sign.

A *tensor module* is a $\mathfrak{gl}_n(\mathbb{R})$-submodule of some tensor product $V^{\otimes r} \otimes (V^*)^{\otimes s}$, $r, s \in \mathbb{N}$.

Theorem B.2 ([19], Theorem 24.4). *For every $\sigma \in \Sigma_r$, let $\Psi_\sigma \in V^{\otimes r} \otimes (V^*)^{\otimes r}$ given by*

$$\Psi_\sigma(v_1 \otimes \cdots \otimes v_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_r) \mapsto \alpha_1(v_{\sigma(1)}) \dots \alpha_r(v_{\sigma(r)}) \quad \forall \alpha_i \in V^*, v_i \in V.$$

They generate the $\mathfrak{gl}_n(\mathbb{R})$-invariant subspace of $V^{\otimes r} \otimes (V^*)^{\otimes r}$. For $r \leq n$, the set of $\{\Psi_\sigma\}$ are linearly independent.
Further, if $r \neq s$, then $(V^{\otimes r} \otimes (V^*)^{\otimes s})^{\mathfrak{gl}_n(\mathbb{R})} = 0$.

Remark B.3. While the generation of the invariants in $V^{\otimes r} \otimes (V^*)^{\otimes r}$ requires a careful analysis which we will not give here, we do want to mention that the nonexistence of invariants in $V^{\otimes r} \otimes (V^*)^{\otimes s}$ for $r \neq s$ is easy: The identity matrix in $\mathfrak{gl}_n$ acts on $V^{\otimes r} \otimes (V^*)^{\otimes s}$ by multiplication with the scalar $r - s$.

Theorem B.4 ([33], Theorem 10). *If $\mathfrak{g}$ is a finite-dimensional, reductive Lie algebra, and V is a finite-dimensional, semisimple $\mathfrak{g}$-module. Then $H^\bullet(\mathfrak{g}, V) = H^\bullet(\mathfrak{g}, V^{\mathfrak{g}})$.*

Lastly, we want to note:

Theorem B.5 ([9], Theorem 2.1.1). *The cohomology ring $H^\bullet(\mathfrak{gl}_n(\mathbb{R}))$ is isomorphic to the exterior algebra*

$$\Lambda^\bullet[\phi_1, \dots, \phi_{2n-1}],$$

where the ϕ_i are generators in degree i.
The inclusion $\mathfrak{gl}(n-1, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ induces a morphism

$$H^q(\mathfrak{gl}(n, \mathbb{R})) \to H^q(\mathfrak{gl}(n-1, \mathbb{R}))$$

which is an isomorphism for $q \leq 2n - 3$.

Remark B.6. Note that our reference states the above theorem in an erroneous way: They state the map induced by the inclusion has a one-dimensional kernel for $q = n$, which, for example, cannot be true when $n = 2$, since the second cohomology vanishes for all $\mathfrak{gl}(n, \mathbb{R})$. They also write that the inclusion only induces an isomorphism in degree $< n$, but their spectral sequence argument actually shows the above, stronger property (see also [34, Cor 4D.3].

[1] We want to thank Jason Schuchardt for this simple proof idea, communicated over math.stackexchange.

Appendix C. The Hochschild-Serre Spetral Sequence for locally convex Lie algebras

In the finite-dimensional setting, the Hochschild-Serre spectral sequence is standard and a proof is laid out in [9, Chapter 1.5.1] and [33]. For general locally convex Lie algebras and continuous cohomology, one generally needs a number of topological assumptions. For example, restriction maps of continuous cochains like $C^q(\mathfrak{g}) \to C^r(\mathfrak{h}, \Lambda^{q-r}(\mathfrak{g}/\mathfrak{h})^*)$ are not necessarily surjective if the subspace $\mathfrak{h}$ is not complemented. We formulate some assumptions which suffice for the setting in this paper:

Theorem C.1. *Let $\mathfrak{g}$ be a complete, barrelled, locally convex, nuclear Lie algebra whose strong dual space $\mathfrak{g}^*$ is complete, $\mathfrak{h} \subset \mathfrak{g}$ a finite-dimensional subalgebra, and A a complete, locally convex space on which $\mathfrak{g}$ acts continuously.*

There is a cohomological spectral sequence $\{E_r^{p,q}, d_r\}$ converging to continuous cohomology $H^\bullet(\mathfrak{g})$ with

$$E_1^{p,q} = H^q\left(\mathfrak{h}, C^p\left(\mathfrak{g}/\mathfrak{h}, A\right)\right),$$

where $C^p(X,Y)$ denotes skew-symmetric, jointly continuous, multilinear maps

$$\underbrace{X \times \cdots \times X}_{p \ times} \to Y,$$

and cohomology is taken with respect to continuous cochains.

This spectral sequence is contravariantly functorial, in the sense that a diagram of continuous Lie algebra morphisms

$$\begin{array}{ccc} \mathfrak{h} & \hookrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{h}} & \hookrightarrow & \tilde{\mathfrak{g}} \end{array}$$

induces linear maps

$$E_r^{p,q}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) \to E_r^{p,q}(\mathfrak{g}, \mathfrak{h})$$

compatible with the differentials for all $p, q, r \geq 0$.

Proof. We define on the continuous cochains $C^\bullet(\mathfrak{g})$ the filtration

$$F^p C^{p+q}(\mathfrak{g}; A) := \{c \in C^{p+q}(\mathfrak{g}, A) : c(X_1, \dots, X_{p+q}) = 0 \text{ when } X_1, \dots, X_{q+1} \in \mathfrak{h}\}.$$

This is an ascending filtration with

$$C^r(\mathfrak{g}, A) = F^0 C^r(\mathfrak{g}, A) \supset \cdots \supset F^r C^r(\mathfrak{g}, A) \supset F^{r+1} C^r(\mathfrak{g}, A) = 0,$$

and

$$dF^p C^{p+q}(\mathfrak{g}; A) \subset F^p C^{p+q+1}(\mathfrak{g}; A).$$

Denote by $\hat{\Lambda}^q$ the functor assigning to a locally convex vector space X the closure of the skew-symmetric tensors in its iterated projective tensor product $X^{\widehat{\otimes}^q}$, see for example [35, Chapter III.7, IV.9].

We have a well-defined map

$$F^p C^{p+q}(\mathfrak{g}, A) \to L(\hat{\Lambda}^q \mathfrak{h} \mathbin{\widehat{\otimes}} \hat{\Lambda}^p \mathfrak{g}/\mathfrak{h}, A), \quad c \mapsto \tilde{c},$$
$$\tilde{c}((h_1 \wedge \cdots \wedge h_q) \otimes [g_1] \wedge \cdots \wedge [g_p])) := c(h_1, \dots, h_q, g_1, \dots, g_p).$$

This map is independent of the choices of representatives g_i by definition of the filtration and it is surjective because finite-dimensional subspaces are always complemented, so $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ as a direct sum of locally convex vector spaces. The kernel equals $F^{p+1}C^{p+q}(\mathfrak{g}, A)$. The image of this map is also indeed contained in the continuous linear maps by continuity of elements in the domain.

Since $\mathfrak{h}$ is finite-dimensional, we trivially have

$$(\mathfrak{h} \mathbin{\widehat{\otimes}} \mathfrak{g}/\mathfrak{h})^* \cong \mathfrak{h}^* \mathbin{\widehat{\otimes}} (\mathfrak{g}/\mathfrak{h})^*.$$

By the assumptions on $\mathfrak{g}$ and A, we may apply [22, Proposition 50.5] twice to find

$$L(\hat{\Lambda}^q\mathfrak{h} \mathbin{\widehat{\otimes}} \hat{\Lambda}^p\mathfrak{g}/\mathfrak{h}, A) \cong L(\hat{\Lambda}^q\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h}, A)) \cong C^q(\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h}, A)).$$

Hence we get an isomorphism of vector spaces

$$F^pC^{p+q}(\mathfrak{g}, A)/F^{p+1}C^{p+q}(\mathfrak{g}, A) \cong C^q\left(\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h}, A)\right).$$

The differential of $C^\bullet(\mathfrak{g}, A)$ descends to the differential of this complex like in the purely algebraic case, so the spectral sequence associated to this filtration indeed has first page:

$$E_1^{p,q} = H^q\left(\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h}, A)\right).$$

The functoriality with respect to Lie algebra pairs $(\mathfrak{g}, \mathfrak{h})$ is analogous to the purely algebraic setting. □

Remark C.2. This spectral sequence in the algebraic setting is generally also phrased with information about the second page if $\mathfrak{h}$ is an ideal. Adapting this to the continuous setting would require stronger assumptions, since this in particular requires commuting the projective tensor product with the cohomology.

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